Distributed schemes for stability and optimality in power networks

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A thesis submitted for the degree of
Doctor of Philosophy

July 2017

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other University. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, except where specifically indicated in the text. This dissertation contains less than 65,000 words including appendices, bibliography, footnotes, tables and equations and has less than 150 figures.

Abstract

The generation, transmission and distribution of electricity underpins modern technology and constitutes a necessary element for our development and economic functionality. In the recent years, as a result of environmental concerns and technological advances, private and public investment have been steadily turning towards renewable sources of energy, resulting in a growing penetration of those in the power network. This poses additional challenges in the control of power networks, since renewable generation is in general intermittent, and a large penetration may cause frequent deviations between generation and demand, which can harm power quality, damage equipment and even cause blackouts.

Load side participation in the power grid is considered by many a means to counterbalance intermittent generation, due to its ability to provide fast response at urgencies. Industrial loads as well as household appliances, such as refrigerators and air-conditions, may respond to frequency deviations by accordingly adjusting their demand in order to support the network. This is backed by the development of relevant sensing and computation technologies.

The increasing numbers of local renewable sources of generation along the introduction of controllable loads dramatically increases the number of active elements in the power network, making traditionally implemented, centralised control difficult and costly. This demonstrates the need for the employment of highly distributed schemes in the control of generation and demand. Such schemes need to ensure the smooth and stable operation of the network. Furthermore, an issue of fairness among controllable loads needs to be considered, such that it is ensured that all load participants share the burden to support the network evenly and with minimum disruption.

In this thesis, we study the dynamic behaviour of power networks within the primary and secondary frequency control timeframes. Using tools from linear and non-linear control and optimisation, we present methods to design distributed control schemes for generation and demand that guarantee stability and fairness in power allocation. Our analysis provides relaxed stability conditions in comparison with current literature and allows the inclusion of practically relevant classes of generation and demand dynamics that have not been considered within this setting, such as of higher order dynamics. Furthermore, fairness in the power allocation between loads is guaranteed by ensuring that the equilibria of the system are solutions to appropriately constructed optimisation problems. It is evident that a synchronising variable is required for optimality to be achieved and frequency is used as such in primary control schemes whereas for secondary frequency control a different synchronising variable is adopted. For the latter case, the requirements of the synchronising feedback scheme have been relaxed with the use of an appropriate observer, showing that stability and optimality guarantees are retained.

The problem of secondary frequency regulation where ancillary services are provided from switching loads is also considered. Such loads switch on and off when some prescribed frequency threshold is reached in order to support the power network at urgencies. To study their behaviour, tools from discontinuous and hybrid systems analysis have been employed. We show that the presence of switching loads does not compromise the stability of the power network and reduces the frequency overshoot, potentially saving the network from collapsing. Furthermore, we explain that when the on and off switching frequencies are equivalent, then arbitrarily fast switching phenomena might occur, something undesirable in practical implementations. As a solution to this problem, hysteresis schemes where the switch on and off frequencies differ are proposed and stability guarantees are provided within this setting.

All our analytic results are distributed and network independent and have been verified with realistic simulations on well accepted benchmarks.

Acknowledgements

First and foremost, I would like to express my deep gratitude to my PhD supervisor and mentor Ioannis Lestas for his guidance and support throughout the last four years. His patience and diligence in work have been of great value for my personal and academic development. His enthusiasm and willingness to discuss about complex issues gave me persistence and inspiration to tackle my research problems. Finally, I appreciate that he gave me the flexibility to pursue my own research interests and useful advices on how I should proceed with my career.

I would also like to thank my colleague and friend Nima Monshizadeh. Our lengthy discussions during the last year have been inspirational to my research. His advices and guidance in both research and career decisions have been of great value to me. Working together as a colleague but most importantly spending time together as a friend has been a privilege and a pleasure.

I am indebted to my former supervisor Jorge Goncalves, who guided me through my first research steps and sparkled my interest in control. His encouragement to continue my studies has been pivotal in choosing to pursue a PhD degree. I would also like to give special thanks to my examiners, Fulvio Forni and David Angeli for proofreading my thesis and suggesting ways to improve it. Their valuable feedback is appreciated.

My sincerest gratitude goes to my colleagues who shaped my work environment and helped me develop both personally and academically. I would like to thank Eoin Devane, who has been the first person of my age to collaborate with, for his valuable role in writing and polishing my first papers and for his help during the first half of my PhD studies. Furthermore, I would like to thank the people in the Power Group for our interesting and inspiring conversations during lunchtime and afternoon tea breaks. Their presence made the lab a nice place to work. I had the privilege to meet very interesting people and exchange insightful ideas and views. In particular, I feel lucky to have met Loizos, my housemate, and Dimitrios and Thilini, who have now left Cambridge, and with whom I hope to stay in touch in the future. During the last year, I had the joy to have a new person in my group, Jeremy, with whom I have enjoyed long discussions over lunch. Finally, I would like to thank the people in the Control group who made me feel welcome since I have moved.

I want to thank my friends in Cambridge for their kind support and good memories. Particularly, I would like to thank Zacharias, whom I had the privilege to become friends with during the last three years, for our interesting discussions at Cafe Nero and our joyful games of tennis. Moreover, I would like to thank Christiana and Olympia, for the great memories, some of which dancing salsa with the former and being in Amsterdam with the latter, and for their support during this time. I want to also express my appreciation to Doros and Christos, for the many nice times we had, either being out or doing barbecue at Christos place and for the joyful time we had in Dublin. Finally, I want to thank all of my friends who although have not been in Cambridge, their support and attitude, and also their visits, have had a positive impact on my time there. In particular, I am grateful to my friends Marios, Vasilis, Yiannis and Andreas for their visits and the unforgettable memories in our trips.

I would like to also express my gratitude to my dear sister, Kalia, for her precious support during this time, her numerous visits to Cambridge and for accompanying me to Las Vegas and Australia. Her kind presence has always been of great value to me.

My beautiful Christina, thank you for your devotion, support and incredible patience during all these years. Your presence has been a continuous source of encouragement and inspiration and the wonderful time we have spent together both in Cambridge and Cyprus, but also in our trips, a joyful memory that I always look back to. I look forward to be able to spent more time with you and do my best to make you happy.

Finally, I would like to thank my beloved parents. I find it difficult to express in words my gratitude to the people who taught me how to express in words. My wholeheartedly thanks for your continuous support, not only during my PhD time but throughout my entire life. I can say with absolute confidence that I would never be able to be at this position without your support and guidance. As a thank you, I dedicate this thesis to you.

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$\mathbb{R}$	: The set of real numbers.
$\mathbb{R}_+$	: The set of nonnegative real numbers, $\mathbb{R}_+ := \{q \in \mathbb{R} \colon q \geq 0\}$.
$\mathbb{C}$	: The set of complex numbers.
$\mathbb{N}$	: The set of natural numbers.
$\mathbb{N}_0$	: The set of natural numbers including zero.
∞	: Positive infinity, also denoted by $+\infty$.
j	: The imaginary unit, $j^2 = -1$.
$\Re$	: The real part of a complex number.
$\Im$	: The imaginary part of a complex number.
$\bar{S}$	: The closure of a set S.
$\bar{co}(S)$	: The convex hull of a set S.
$\|A\|$	: Number of elements within set A, also referred as cardinality of A.
$A \subset B$	: A is subset of B, i.e. if $x \in A \Rightarrow x \in B$.
$\emptyset$	: The empty set.
$[a, b]$	$:= \{x \in \mathbb{R}^n \colon a \leq x \leq b\}$. The closed interval between any $a, b \in \mathbb{R}^n$.
(a, b)	$:= \{x \in \mathbb{R}^n \colon a < x < b\}$. The open interval between any $a, b \in \mathbb{R}^n$.
$x(t)$	: The value of a function $x : \mathbb{R}_+ \to \mathbb{R}^n$ at time $t \geq 0$.
$\frac{dx}{dt}(t)$	: The time-derivative of a continuously differentiable function $x : \mathbb{R}_+ \to \mathbb{R}^n$ evaluated at time $t \geq 0$.
x_k	: The kth element of vector x.
$A_{i,j}$	: The element in the ith row and jth column of matrix A.
$[q]_a^b$	: The saturation $[q]_a^b := \max\{\min\{q, b\}, a\}$.
g'	: The first-derivative of a differentiable function of one variable g.
g^{-1}	: $g^{-1}(w)$ denotes the preimage of the point w under the function g, defined as $g^{-1}(w) := \{q \colon g(q) = w\}$.
I_n	: The identity matrix of size $n \times n$.
Q^T	: The transpose of a matrix Q.
Q^{-1}	: The inverse of the invertible matrix Q.
$\\|x\\|$	: The norm of vector x.
$B(x, \delta)$	: Ball of radius $\delta > 0$ centered at x, $B(x, \delta) = \{q : \\|q - x\\| \leq \delta\}$.

Chapter 1

Introduction

This chapter contains a brief introduction on power network control and optimisation and a discussion on its potential impact which motivates research on the topic. The main contribution and structure of this thesis are then described in order to enhance its readability.

1.1 Motivation of this work

The robust and reliable generation, transmission and distribution of electric power is an integral part of our technological evolution which is the foundation of modern civilisation. Since their first creation, in the late 19th century, power networks have been continuously evolving. It is expected that their development will continue within the next decades and shall be driven by two major factors, the increase in penetration of renewable sources of generation and the development of sensing, computation and communication technologies.

A growing attention is paid on renewable sources of power generation, due to environmental concerns driving popular opinion and influencing political agentas, with investments of $285.9 billion within 2015 amounting to an addition of 118GW in wind and solar photovoltaics only [3]. Furthermore, the Paris Agreement, signed by 195 countries, that went into effect on November 2016 has been described as a decisive incentive for fossil fuel divestment [4], strengthening the expectations for further investments on renewables. However, it is known that renewable generation is in general intermittent and a large penetration would cause large, fast fluctuations in the generated power, potentially resulting to severe frequency deviations which can degrade power quality and load performance, damage equipment and even cause blackouts.

A possible solution to this problem comes from load side participation. Controllable loads may provide fast response when necessary, counterbalancing intermittent generation. Such loads may be large, such as industrial loads, or small, such as smart appliances (e.g. refrigerators, heat pumps, space heater, air conditions) and power storage devices. The development of controllable loads and their introduction in the power network has been supported by the rapid progress in communication, sensing and computer infrastructures along advanced power electronics such as phasor measurement units and smart meters.

The expected growth of controllable loads and local, small scale generation resources within the power network will make it far more complex. This motivates the study of power network behaviour, in order to better understand the rich dynamics that result from the interconnection and interaction of a population of heterogeneous dynamical systems. A large scale integration of such active elements within the network requires the study and development of distributed schemes for the control of local generation and demand such that plug and play operation with minimum communication complexity will be possible. Moreover, stability guarantees are required to ensure that the highly distributed nature of loads and their interaction with existing control schemes in the network will not result to undesirable and potentially harmful behaviours.

An issue of fairness in the power allocation is also raised if controllable loads are to participate in the power network. This can be achieved by ensuring that the equilibrium allocation of loads is a solution to an appropriately constructed optimisation problem that guarantees the desired fairness in allocation.

This thesis attempts to address the above by studying the power network behaviour within the primary and secondary frequency control timeframes. The main objectives considered are to balance generation and demand and ensure that the frequency remains close to its nominal value and synchronises while also taking into account fairness considerations. An additional requirement within secondary frequency control is to ensure that frequency returns to its nominal value. We study the interaction of multiple interconnected active components, such as generators, controllable loads and storage units, and try to obtain distributed conditions such that all the above objectives are achieved.

The contributions presented in this thesis, along a description of each chapter's content, are outlined in the following section.

1.2 Contribution and structure of this thesis

The structure of this thesis along its main contributions in literature are summarised below.

Chapter 2 contains a brief introduction on the topic of smart grids, outlining the potential benefits of their deployment and presenting various relevant recent research and deployment projects. A model that describes the power network dynamics is derived, explicitly stating and explaining all assumptions made. The main objectives of primary, secondary and tertiary frequency control are then discussed, explaining how each control layer achieves their accomplishment and how optimality considerations are also taken into account. Finally, we review the literature on primary and secondary frequency control, discussing its historical development on each control layer and referring to recent studies where demand side management has been considered, taking both stability and optimality issues into account.

Chapter 3 considers several key concepts of stability and optimality that are later used within this thesis. We first consider ordinary differential equations and describe and discuss notions of solutions and stability, presenting the important results of Lyapunov's direct method and Lasalle's invariance principle. We then define solutions and stability notions for systems with discontinuous and hybrid dynamics, reviewing some important results on existence and uniqueness of solutions and two key invariance principles. Intuitive remarks that aid to better understand the relevant concepts are made throughout the chapter. Finally, we present the KKT conditions which are used to establish optimality of equilibria, and discuss how they can be extended to include non differentiable cost functions and constrains by using subgradient techniques.

The main technical content of the thesis is divided into two parts. Part I contains Chapters 4 and 5 and considers primary frequency regulation with load side participation. Part II studies secondary frequency regulation where also load side participation is considered and contains Chapters 6 and 7.

In Chapter 4 we present a method to design distributed generation and demand control schemes for primary frequency regulation in power networks that guarantee asymptotic stability and ensure fairness of allocation. The main stability condition imposed is the passivity of net power supply variables. We also provide explicit steady state conditions on a general class of generation and demand control dynamics that ensure that the corresponding equilibria solve an appropriately constructed network optimization problem that guarantees fairness in power allocation and generation-demand balance. Moreover, we discuss how various classes of dynamics used in recent studies fit within our framework and show that this allows for less conservative stability and optimality conditions. Furthermore, this framework includes dynamics that have not been considered in literature within this context, such as that of high order turbine governor dynamics.

In Chapter 5 we consider the problem of designing distributed generation and demand control schemes that provide ancillary service in primary frequency regulation such that stability and fairness in the power allocation can be guaranteed. It is desired that load side participation in frequency control is activated only at urgencies, where the frequency exceeds prescribed thresholds, so as to avoid frequent intervention in the operation of loads. This, however, leads to nonlinear control schemes where the derivatives of the vector fields are discontinuous. Within this chapter, we analyse and investigate how stability and optimality may be ensured when such dynamics are considered. We use subgradient methods to derive decentralised conditions that ensure optimality of the equilibrium points and discuss how the stability results of Chapter 4 also apply within this context.

The results presented in Part I are validated with simulations on realistic bus systems.

Part II focuses on the study of secondary frequency control where again load side participation is considered.

Chapter 6 presents a framework for the design of distributed generation and demand control schemes for secondary frequency regulation in power networks such that stability and an economically optimal power allocation can be guaranteed. A dissipativity condition is imposed on net power supply variables to provide stability guarantees. Furthermore, decentralized steady state conditions on the generation and controllable demand are provided, such that economic optimality is achieved. We discuss how various classes of dynamics used in recent studies fit within our framework and give examples of higher order generation and controllable demand dynamics that can be included within our analysis. In case of linear dynamics, we discuss how the proposed dissipativity condition can be efficiently verified using an appropriate linear matrix inequality. Moreover, it is shown how the requirement for demand measurements in the employed controller may be relaxed by the addition of a suitable observer layer. The efficiency and practicality of the proposed results are demonstrated with a simulation on the Northeast Power Coordinating Council (NPCC) 140-bus system.

In Chapter 7 we consider the problem of secondary frequency regulation where ancillary services are provided via load-side participation. In particular, we consider on-off loads that switch when prescribed frequency thresholds are exceeded in order to assist existing secondary frequency control mechanisms. We show that system stability is not compromised despite the switching nature of the loads. However, we show that when the on and off frequencies are equal that loads are prone to transient arbitrarily fast switches, a phenomenon known as Zeno behaviour, which limits their practicality. As a solution to this problem, we propose a hysteresis on-off policy, where the on and off frequencies differ, and provide stability guarantees in this setting.

Finally, Chapter 8 provides a summary of the contribution along intuitive comments that allow the accurate interpretation of the main results. Furthermore, we present suggestions and ideas on how to possibly extend the presented work.

There are various ways in which the main contribution chapters, Chapters 4–7, connect based on their application and methodology. These are summarised below:

- All chapters consider distributed approaches for feedback control design of generation and controllable demand dynamics and provide network independent results.

- The analysis in all chapters considers the design of schemes such that stability is guaranteed. Furthermore, within Chapters 4–6 conditions for fairness in the power allocation are also provided.

- The main contribution of Chapter 5 lies on the fairness conditions provided. In contrast, Chapter 7 only focuses on the stability analysis of its considered system. The contributions within Chapters 4 and 6 are important in terms of both stability and optimality.

- Chapters 4 and 5 consider primary frequency regulation with load side participation.

- Chapters 6 and 7 consider secondary frequency regulation with load side participation using tools from nonlinear analysis. The analysis in Chapter 6 focuses on systems with continuous dynamics whereas Chapter 7 takes into account systems with discontinuous dynamics.

- Chapters 4, 5 and 6 make use of passivity related tools for their stability analysis.

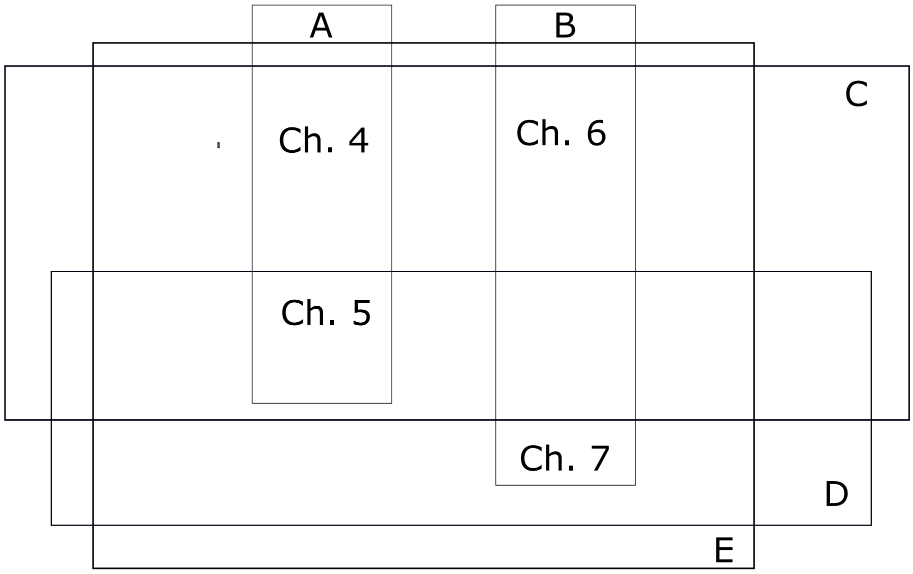

Figure 1.1. Schematic representation of the connections among Chapters 4–7 based on the main application topics and analysis tools. The letters represent: A. Primary frequency control, B. Secondary frequency control, C. Passivity analysis techniques, D. Analysis of discontinuities - loads providing ancillary service, E. Nonlinear analysis.

- Both Chapters 5 and 7 consider the use of loads to provide ancillary services in power networks. Moreover, discontinuities are taken into account in both chapters analyses. However, the analysis in Chapter 5 considers discontinuities in the derivative of the vector field whereas in Chapter 7 the discontinuities considered are in the vector field itself. In contrast, the analysis and discussion in Chapters 4 and 6 does not take into account any form of discontinuity[1].

The aforementioned connections among the chapters are schematically presented in Figure 1.1.

The work within Chapters 2 and 4 - 7 is based upon the following publications which have been produced during the Ph.D. course [5], [6], [7], [8], [9], [10], [11], [12], [13].

Peer Reviewed Journals:

1. A. Kasis, N. Monshizadeh, and I. Lestas "Secondary frequency control with switching load side participation in power networks", under review at IEEE Transactions on Power Systems.

[1]Note that the Lipschitz continuity condition considered in Chapters 4 and 6 includes non continuously differentiable vector fields. However, the discussion and examples provided focus on continuously differentiable dynamics.

2. A. Kasis, N. Monshizadeh, E. Devane, and I. Lestas "Stability and optimality of distributed secondary frequency control schemes in power networks", to appear at IEEE Transactions on Smart Grids

3. A. Kasis, E. Devane, C. Spanias and I. Lestas, "Primary frequency regulation with load-side participation. Part I: stability and optimality", IEEE Transactions on Power Systems, 2016

4. E. Devane, A. Kasis, M. Antoniou and I. Lestas, "Primary frequency regulation with load-side participation. Part II: beyond passivity approaches", IEEE Transactions on Power Systems, 2016

International Conference Proceedings:

1. A. Kasis, N. Monshizadeh, and I. Lestas "Secondary frequency control with on-off load side participation in power networks", Proceedings of the 56th IEEE Conference on Decision and Control

2. A.Kasis, E. Devane, and I. Lestas, "Primary frequency regulation in power networks with ancillary service from load-side participation", Proceedings of IFAC 2017 World Congress

3. A.Kasis, E. Devane, and I. Lestas, "Stability and optimality of distributed schemes for secondary frequency regulation in power networks", Proceedings of the 55th IEEE Conference on Decision and Control, December 2016

4. A. Kasis, E. Devane, and I. Lestas, "On the stability and optimality of primary frequency regulation with load-side participation", Proceedings of the 54th IEEE Conference on Decision and Control, December 2015

Book Chapters:

1. E. Devane, A. Kasis, C. Spanias, M. Antoniou, and I. Lestas, "Distributed frequency control and demand-side management", in Smarter Energy: From Smart Metering to the Smart Grid, Institution of Engineering and Technology, 2016

Chapter 2

Literature review

In this chapter we discuss the potential benefits of smart grids deployment, which motivates research on the topic, and provide examples of successful implementations along a forecast for large future investments for their development. A model that characterizes power network behaviour under nominal operation conditions, used within the rest of this manuscript, is derived and intuitive explanations on all the assumptions taken are provided along useful remarks on their rational. Moreover, the current operation of the power network is reviewed and the main objectives and function of frequency control layers are explained. Furthermore, we discuss current approaches in the literature such that stability and optimality guarantees are obtained within both primary and secondary control, providing intuitive remarks on the analysis.

2.1 Smart grids

In this section we briefly discuss the concept and potential benefits of smart grids and provide various examples of research and development projects that have been carried out. The potential benefits of developing a smart grid are outlined in the following subsection.

2.1.1 Potential benefits

A smart grid is expected to make use of information provided from stations, substations and consumers to optimise its decisions to become more efficient and reliable. The potential of a successfully implemented smart grid is summarised below.

Using provided information and advanced control methods such as state estimation and forecasting [14], a grid operator may be able to detect when an element is under-performing or is near its limits. Hence, generation and demand may be adapted accordingly to limit or even avoid the use of particular elements of the grid, such as power lines or generators. Therefore, the power network reliability will be strengthened. Furthermore, the network elements (e.g. cables, transformers in substations) condition will possibly be available in real time. Hence, the current periodic checks will become obsolete and maintenance operations will be performed only when necessary. This could prevent unnecessary replacements in equipment and eliminate the faults caused by non properly maintained elements. From an economic point of view, this could drop maintenance costs with a potential effect on electricity prices.

The implementation of a smart grid may also allow for a high penetration of renewable sources of energy in the power network. It is known that renewable generation causes fast fluctuations in generated power that cannot be counterbalanced by slow conventional power plants. Hence, a high penetration of renewable generation may compromise network's stability. A smart grid, using tools such as forecasting, stochastic control and load management would be able to adapt to fluctuating generation which would be a key element for an energy sustainable future.

A very promising aspect of smart grids is the potential for demand control which is the adjustment of load consumption for network support provision. A power grid could shift loads from periods of high demand to periods of low demand, reducing the use of the least efficient generators. This could decrease the peak demand and therefore the need for additional power infrastructure to support it, which could significantly decrease electricity bills for consumers. Controllable loads may also be a solution towards a high penetration of renewable sources of energy within the power grid, providing fast response to intermittent generation.

Finally, a smart grid may enable market mechanisms to control demand by providing varying energy prices to consumers, that will respond to supply and demand rules. With the current communication infrastructure, the price could be easily accessible by consumers that would adapt their consumption accordingly. This would enhance both suppliers and consumers flexibility to adopt various operation strategies. Using the provided information, a network operator will be able to determine a fair price for an energy unit, taking into account the system's current state and infrastructure.

2.1.2 Recent research projects and deployments

The aim for energy sustainability and the need for a more efficient power system lead significant investments in the research and development of smart grids. This trend is expected to grow with time, as shown on Table 2.1.

Country/Region	Investment Forecast (€)	Investment for Current Development (€)
European Union	56 billion by 2020 500 billion by 2030	384 million
USA	238-334 billion by 2030	4.9 billion
China	71 billion	5.1 billion
South Korea	16.8 billion by 2030	580 million

Table 2.1. Current and forecasted investments on smart grids in 4 large economies [2].

The growing attention on smart grid technologies is reflected on numerous related research and development projects. The most notable are discussed below.

The Modern Grid Initiative (MGI) [15] program by the National Energy Technology Laboratory (NETL) aims to develop a detailed vision and plan for the transformation of the US grid system towards a modern grid where smart technologies, including demand response and distributed energy resource, will be employed. Furthermore, the practical benefits of using controllable devices have been demonstrated by several field tests. For instance, demand response schemes, together with various other smart grid technologies, have been successfully trialled in Ontario, Canada by the company Hydro One [16]. Additionally, a demand response scheme was successfully implemented by the Pacific Northwest National Laboratory in 2006-07, with 112 homes changing their electric consumption for water and space heating using price signals [17].

Based upon the promising results from field tests such as these, several recent initiatives and smart grid related projects encourage the use of smart appliances and demand-side management. For example, the IntelliGrid project devised by the Electric Power Research Institute (EPRI) in the US provides recommendations to maximise utilisation using existing infrastructure, and demand response is included as a tool to achieve this objective. Several aspects of its proposed architecture have been implemented by utility companies and there are plans for further demonstration projects which include the incorporation of demand response schemes based upon price signals [18]. Moreover, Grid2030 [19] is a vision statement for the US electrical system, proposed by various important stakeholders, describing a pathway for the future evolution of the power grid in terms of generation, transmission, distribution, and storage with demand response included. The GRID4EU project [20], mainly funded by the European Commission with the cooperation of 6 distribution companies, aims for the development and study of smart grid technologies. It specified targets on the use of more renewable energy sources, connected to distribution networks, to encourage customer participation in electricity markets and to develop means of demand-side management.

There have been several further smart grid deployments over the past decade throughout the world. Their success has driven more stakeholders to consider adapting their networks to modern standards. An example of a successful smart grid implementation is Enel's project, developed in Italy and completed in 2005. It currently delivers annual savings of 500 € million with an investment of 2.1 € billion [21]. Further examples include the implementation of smart metering in Boulder, Colorado by XCEL Energy [22] and the Smart Energy Collective project in Netherlands [23].

2.2 Power network modelling

This section contains a description of the dynamics that govern a power network. Within it, we derive the models that we use within this manuscript, explicitly stating and justifying all assumptions taken. The models follow from [24] and [25]. Additional interesting studies on power system modelling are [26, 27].

2.2.1 Power network structure - main quantities

We begin by describing the structure of a power network which, following the analysis in [24] and [25], may be described as a graph interconnection of buses (nodes) and power lines. Within this representation, the following four quantities are associated to each bus k

- V_k - voltage magnitude
- θ_k - voltage angle
- P_k - net active power (algebraic sum of generation and load)
- Q_k - net reactive power (algebraic sum of generation and load)

Depending on which of the above quantities are known and which calculated, buses are divided into the following three categories:

- PQ bus: P_k and Q_k are specified; V_k and θ_k are calculated
- PV bus: P_k and V_k are specified; Q_k and θ_k are calculated
- $V\theta$ bus: V_k and θ_k are specified; P_k and Q_k are calculated

where PQ buses normally correspond to load buses without voltage control, PV buses represent generation buses with voltage control and $V\theta$ buses represent slack buses. For convenience, we shall refer to PQ, PV and $V\theta$ buses as load, generation and slack buses respectively.

2.2.2 Power flow modelling

In this subsection a π-model with the most common elements used for power flow analysis will be considered. Within the analysis below, symmetrical three phase conditions are assumed. A general distributed power flow model is characterized by the following series and shunt elements per each phase

- r : series resistance per km (Ω/km)
- x : series reactance per km (Ω/km)
- b : shunt susceptance per km (S/km)
- g : shunt conductance per km (S/km)

as depicted in Figure 2.1. Voltage and current variation with time and distance may be derived from this model by using telegrapher's equations [28], a set of two coupled partial differential equations. The representation of Figure 2.1 has a π-model equivalent for medium sized lines of up to $200km$ [25] which is significantly more convenient for analysis purposes. Its derivation follows from relatively simple circuit analysis[1]. The π-model, between buses k and m is depicted on Figure 2.2. It is characterized by the series impedance z_{km} and shunt admittance y_{km}^{sh} described by

$$z_{km} = r_{km} + jx_{km} \text{ (series impedance } (\Omega)), \tag{2.1a}$$

[1]The interested reader is referred to [25], [29].

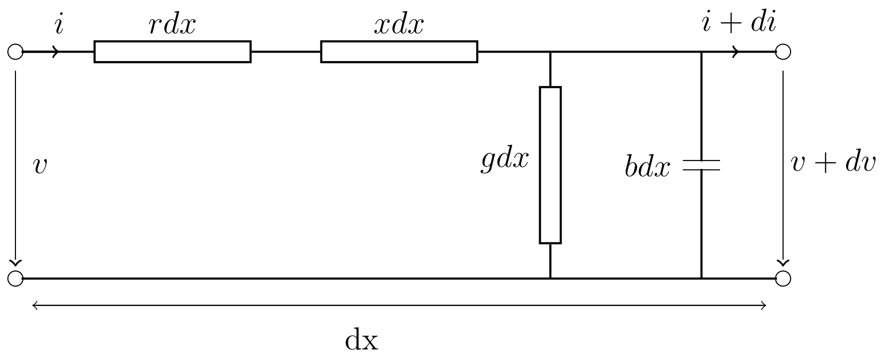

Figure 2.1. Equivalent circuit of a line element of length dx, with input and output current-voltage pairs (i, v) and $(i + di, v + dv)$.

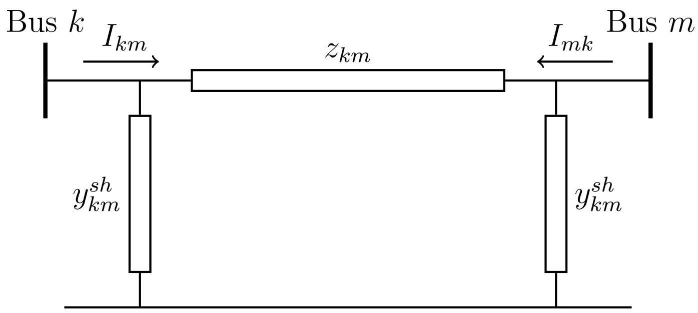

Figure 2.2. Lumped circuit model (π-model) representation of a transmission line from bus k to bus m.

$$y_{km}^{sh} = g_{km}^{sh} + jb_{km}^{sh} \text{ (shunt admittance } (S)). \tag{2.1b}$$

Note that within the π-model, it is assumed that lines are homogeneous, i.e. line parameters have equal values along their lengths. This allows to say that $y_{km}^{sh} = y_{mk}^{sh}$.

To formulate the node admittance matrix, we shall make use of the series admittance and complex voltages at each bus, defined as

$$y_{km} = z_{km}^{-1} = g_{km} + jb_{km}, \tag{2.2a}$$

$$E_i = V_i e^{j\theta_i}, \ i = k, m, \tag{2.2b}$$

where the series conductance and susceptance g_{km} and b_{km} are respectively defined by

$$g_{km} = \frac{r_{km}}{r_{km}^2 + x_{km}^2}, \tag{2.3a}$$

$$b_{km} = -\frac{x_{km}}{r_{km}^2 + x_{km}^2}. \tag{2.3b}$$

Note that for physical transmission lines, both series resistance r_{km} and series reactance x_{km} are positive and hence g_{km} is positive and b_{km} negative.

The complex currents from bus k to bus m and from bus m to bus k, I_{km} and I_{mk}, can then be expressed by

$$\begin{aligned} I_{km} &= y_{km}(E_k - E_m) + y_{km}^{sh} E_k, \\ I_{mk} &= y_{km}(E_m - E_k) + y_{km}^{sh} E_m, \end{aligned} \tag{2.4}$$

and hence the admittance matrix is described by

$$\begin{bmatrix} I_{km} \\ I_{mk} \end{bmatrix} = \begin{bmatrix} y_{km} + y_{km}^{sh} & -y_{km} \\ -y_{km} & y_{km} + y_{km}^{sh} \end{bmatrix} \begin{bmatrix} E_k \\ E_m \end{bmatrix}. \tag{2.5}$$

In the analysis that follows, we make the assumption that shunt conductance is negligible and thus $y_{km}^{sh} \approx jb_{km}^{sh}$. Hence the complex power transferred between buses k and m, denoted by S_{km}, may be expressed by

$$S_{km} = E_k I_{km}^* = y_{km}^* V_k e^{j\theta_k}(V_k e^{-j\theta_k} - V_m e^{-j\theta_m}) - jb_{km}^{sh} V_k^2. \tag{2.6}$$

Furthermore, S_{km} may be expressed in terms of its real and imaginary components based on $S_{km} = P_{km} + jQ_{km}$, where P_{km} and Q_{km} respectively represent real and reactive power transfers. Expressions for P_{km} and Q_{km} may be obtained by identifying the real and imaginary parts from equation (2.6) as follows

$$P_{km} = V_k^2 g_{km} - V_k V_m g_{km} \cos\theta_{km} - V_k V_m b_{km} \sin\theta_{km}, \tag{2.7}$$

$$Q_{km} = -V_k^2 (b_{km} + b_{km}^{sh}) + V_k V_m b_{km} \cos\theta_{km} - V_k V_m g_{km} \sin\theta_{km}, \tag{2.8}$$

where $\theta_{km} = \theta_k - \theta_m$.

The use of Kirchoff's current law (KCL) yields the following expression for the current at bus k

$$I_k + I_k^{sh} = \sum_{m \in \Omega_k} I_{km}, \tag{2.9}$$

where I_k is the net current injection from generators and loads, I_k^{sh} the current injection from shunts, Ω_k the set of buses adjacent to k excluding k, and m some bus adjacent to k.

With simple algebraic manipulations, similar to the ones carried to obtain equations (2.7) and (2.8), the net power injections can be obtained as follows

$$P_k = V_k \sum_{m \in \bar{\Omega}_k} V_m (G_{km} \cos\theta_{km} + B_{km} \sin\theta_{km}), \tag{2.10}$$

$$Q_k = V_k \sum_{m \in \bar{\Omega}_k} V_m (G_{km} \sin\theta_{km} - B_{km} \cos\theta_{km}), \tag{2.11}$$

where $\bar{\Omega}_k = \Omega_k \cup \{k\}$, i.e. the set of buses adjacent to k, including k, and the variables G_{km}, B_{km} are the real and imaginary parts of admittance Y_{km}, defined by (2.12) below,

$$\begin{aligned} Y_{km} &= G_{km} + jB_{km}, \\ Y_{km} &= \begin{cases} -y_{km}, \ k \neq m \\ y_k^{sh} + \sum_{m \in \Omega_k} (y_{km}^{sh} + y_{km}), \ k = m, \end{cases} \end{aligned} \tag{2.12}$$

where y_k^{sh} denotes the shunt admittance at node k.

2.2.3 Decoupling active and reactive power flows

For transmission systems, a strong coupling is normally observed between the variable pairs P, θ and Q, V. This property allows the decoupling of real and reactive power, which drastically simplifies the analysis. To make this simplification, we assume that series resistances and shunt admittances are negligible, which simplifies equations (2.7) and (2.8) to

$$P_{km} = \frac{V_k V_m}{x_{km}} \sin\theta_{km}, \tag{2.13}$$

$$Q_{km} = \frac{V_k^2 - V_k V_m \cos\theta_{km}}{x_{km}}, \tag{2.14}$$

where x_{km} is the series reactance of the line.

Note that perfect decoupling is achieved when $\theta_{km} = 0$ and that the above approximation is valid when θ_{km} is small, i.e. at the usual range of operating conditions there is a strong coupling between active power and voltage angle as well as between reactive power and voltage magnitude.

2.2.4 Modelling of generation dynamics

A thorough analysis of generation dynamics is complex and beyond the scope of this thesis. We refer the interested reader to [25, Chapter 11], from where examples will be borrowed in the following chapters to enhance the discussion on the contribution of this work.

2.2.5 Swing equation derivation

The swing equation describes power angle evolution by a second order differential equation, demonstrating power angles dependence on the electrical and mechanical power at a given bus. Below, we present its derivation that follows from [25, Chapter 5] and makes use of the following quantities,

- δ_m : power angle
- ω_m : rotor shaft velocity (rad/s)
- ω_{sm} : synchronous speed, $\omega_m = \omega_{sm}$ at steady state
- J : total moment of inertia of both the turbine and generator rotors $(kg\ m^2)$
- D_d : damping coefficient (Nms), accounts for the mechanical rotational loss due to friction and windage
- τ_t : torque produced by the turbine (Nm)
- τ_e : electromagnetic torque (Nm)
- P_m : net shaft power input (W)
- P_e : electrical power (W)

To derive the swing equation, we make use of Newton's second law applied on the rotor shaft as follows

$$J\frac{d\omega_m}{dt} + D_d\omega_m = \tau_t - \tau_e, \tag{2.15}$$

where the rotor frequency is expressed as the synchronous speed plus a deviation $\Delta\omega = \frac{d\delta_m}{dt}$ as shown below

$$\omega_m = \omega_{sm} + \frac{d\delta_m}{dt}. \tag{2.16}$$

Moreover, the mechanical torque τ_m is equal to the turbine torque minus the rotational losses at steady state as described below

$$\tau_m = \tau_t - \omega_{sm} D_d. \tag{2.17}$$

Substituting (2.16) and (2.17) into (2.15) and multiplying both sides by ω_{sm} yields

$$J\omega_{sm}\frac{d^2\delta_m}{dt} + \omega_{sm} D_d \frac{d\delta_m}{dt} = \omega_{sm}\tau_m - \omega_{sm}\tau_e, \tag{2.18}$$

which can be expressed in terms of the mechanical and electrical power, using the fact that power is the product of torque and angular velocity ($P = \omega\tau$), as follows

$$J\omega_{sm}\frac{d^2\delta_m}{dt} + \omega_{sm} D_d \frac{d\delta_m}{dt} = \frac{\omega_{sm}}{\omega_m} P_m - \frac{\omega_{sm}}{\omega_m} P_e. \tag{2.19}$$

Finally, using the fact that in general $\omega_m \approx \omega_{sm}$, and substituting $M_m = J\omega_{sm}$ (rotor's angular momentum at synchronous speed) and $D_m = \omega_{sm} D_d$ results to the following expression

$$M_m \frac{d^2\delta_m}{dt} + D_m \frac{d\delta_m}{dt} = P_m - P_e, \tag{2.20}$$

which is called the swing equation.

2.2.6 Simplified power network dynamics

Using equations (2.13)–(2.14) and (2.20), derived within the previous sections, we are now in a position to describe the power network dynamics considered within this manuscript, explicitly referring to all additional assumptions made.

We consider power networks described by a connected graph (N, E) where $N = \{1, 2, \ldots, |N|\}$[2] is the set of buses and $E \subseteq N \times N$ the set of transmission lines connecting the buses. It is assumed that the network consists of buses with and without inertia. Since generators always have inertia, it is reasonable to assume that generation buses have non-zero inertia and nontrivial generation dynamics where load buses may have zero inertia. Within the manuscript, buses with inertia shall be referred as generation buses and buses without inertia as load buses[3]. Let $G =$

[2]We remind that for any set A, $|A|$ denotes its cardinality.

[3]We acknowledge that is possible for loads to also contribute to a bus inertia. Therefore, there exist load buses with non-zero inertia. These cases are included in the analysis presented in this thesis but will not be further discussed. We adopt the simpler terms generation and load buses to denote buses with and without inertia for simplicity in presentation.

ω_j	frequency at bus j
η_{ij}	power angle difference between bus i and bus j
p_j^M	mechanical power injection at bus j
d_j^c	controllable load at bus j
d_j^u	uncontrollable frequency dependent load and generation damping at bus j
p_{ij}	power transfer from bus i to bus j
B_{ij}	line susceptance between buses i and j
p_j^L	step change in uncontrollable demand at bus j

Table 2.2. Notation used in the system model (2.21). Note that variables ω_j, p_j^M, d_j^c, d_j^u, p_j^L denote deviations from corresponding nominal values.

$\{1, 2, \dots, |G|\}$ and $L = \{|G| + 1, \dots, |N|\}$ be the sets of generation and load buses such that $|G| + |L| = |N|$. Furthermore, we use (i, j) to denote the link connecting buses i and j and assume that the graph (N, E) is directed with arbitrary direction, so that if $(i, j) \in E$ then $(j, i) \notin E$. For each $j \in N$, we use $i : i \to j$ and $k : j \to k$ to denote the sets of buses that are predecessors and successors of bus j respectively. It is important to note that the form of the dynamics in (2.21) below is unaltered by any change in the graph ordering.

Within the analysis that follows, we shall assume complete decoupling between reactive power/voltage and power/voltage angle following the analysis in Section 2.2.3 and also constant voltages and negligible line resistances. Such assumptions are well justified, particularly at higher voltages in transmission networks or when tight voltage control is possible. We define $B_{ij} = \frac{V_i V_j}{x_{ij}}$ and with slight abuse of notation we shall refer to it as line susceptance, noting its connection with (2.3b) at unit voltage magnitudes and zero resistance.

Therefore, the following assumptions are made for the network:
1) Bus voltage magnitudes are $|V_j| = 1$ p.u. for all $j \in N$.
2) Lines $(i, j) \in E$ are lossless and characterised by their susceptances $B_{ij} = B_{ji} > 0$.
3) Reactive power flows do not affect bus voltage phase angles and frequencies.

We use the swing equations (2.20) to describe the rate of change of frequency at generation buses, while power must be conserved at each of the load buses. Note that the term $D_j \omega_j$ in (2.20) is included within generation and demand dynamics. This motivates the following system dynamics,

$$\dot{\eta}_{ij} = \omega_i - \omega_j, \ (i, j) \in E, \tag{2.21a}$$

$$M_j\dot{\omega}_j = -p_j^L + p_j^M - (d_j^c + d_j^u) - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij},\ j \in G, \tag{2.21b}$$

$$0 = -p_j^L - (d_j^c + d_j^u) - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij},\ j \in L, \tag{2.21c}$$

$$p_{ij} = B_{ij} \sin \eta_{ij} - p_{ij}^{nom},\ (i,j) \in E. \tag{2.21d}$$

In system (2.21) the time-dependent variables p_j^M, ω_j and d_j^c represent, respectively, deviations from a nominal value[4] of the mechanical power injection to the generator bus j, and the frequency and controllable load present at any bus j. The quantity d_j^u is also a time-dependent variable that represents the uncontrollable frequency-dependent load and generation damping present at bus j. Furthermore, the quantities η_{ij} and p_{ij} are time-dependent variables that represent, respectively, the power angle difference, and the deviation from the nominal value, p_{ij}^{nom}, of the power transmitted from bus i to bus j. The constant $M_j > 0$ denotes the generator inertia.

Within this manuscript, we shall study the response of system (2.21) at a step change in the uncontrollable demand p_j^L at each bus j. Note that for convenience, all the variables used in (2.21) are displayed on Table 2.2.

Remark 2.1 *The variables $p_j^M, j \in G$ and $d_j^c, d_j^u, j \in N$ are outputs of dynamical systems that are defined in each of Chapters 4–7 and will be seen to be 'closing the loop' for system* (2.21). *Therefore, from a control perspective, system* (2.21) *can be seen as a system with input $p_j^L, j \in N$ and states $\eta_{ij}, (i,j) \in E$, $\omega_j, j \in G$ and the internal states of generation and demand dynamics. The inclusion of demand dynamics allows the frequency at load buses, $\omega_j, j \in L$, to be implicitly defined. Hence, the system consisting of the interconnection of* (2.21) *and generation-demand dynamics will have well defined solutions, despite the presence of the direct algebraic equations* (2.21c).

2.3 Overview of frequency control

Frequency control is of high importance for the safe and reliable operation of the power network. It has been traditionally implemented from the generation side via

[4]A nominal value is defined as an equilibrium of (2.21) with frequency equal to 50 Hz (or 60 Hz).

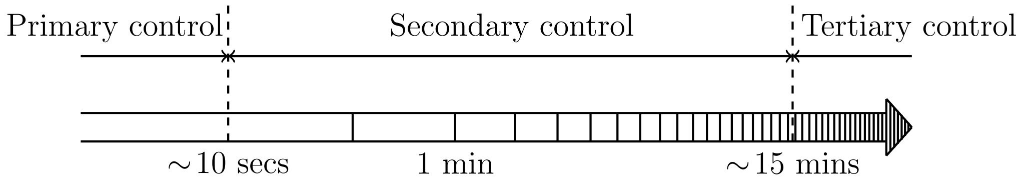

Figure 2.3. Typical timescales of the main control schemes in the power grid. Typical values taken from [1]. Note that time is indicated on a logarithmic scale.

the primary and secondary control stages. Furthermore, a tertiary control stage has been responsible to minimise the generation costs along certain operation constrains.

The main objective of a power gird is to balance generation and demand which is achieved by the primary, secondary and tertiary frequency control stages, named from their different timescales, as depicted on Figure 2.3. In particular, the primary control stage acts within tens of seconds from a change in demand and adapts generation to match demand resulting to a deviation in frequency. This deviation is corrected by secondary frequency control action, acting from tens of seconds up to 15 minutes. The tertiary control, acting from about 15 minutes to a few hours controls generator set points to match predicted demand profiles, based on aggregate users consumption.

This section offers a brief description of the operation of each of these separate schemes. A more in-depth description of each of these control stages and how they are implemented in practice can be found in various textbooks, such as [25, 26, 30].

2.3.1 Primary frequency control in the power grid

Primary frequency control is responsible to balance changes in generation/demand on short timescales. Such disturbances can result from a sudden loss of generation capability or the unexpected introduction of additional load in the power grid. The function of primary control is to appropriately adjust generation profiles such that the aforementioned disturbances are met and hence ensure the desired generation/demand balance. A result of this action is a deviation on the grid frequency from its nominal value. Primary control action aims to regulate frequency within permitted tolerance from nominal since large deviations could harm power quality and even cause blackouts. Such schemes are usually distributed in nature, with generators using local frequency as control input to regulate their output.

A typical example of a primary frequency control scheme is droop control, which is usually a proportional feedback scheme that gets activated when the frequency deviation is sufficiently large. Following the swing equation (2.20) and power system dynamics (2.21), it can be seen that an increase in demand or a loss in generation of magnitude Δp^L, results to a steady state frequency deviation from nominal $\Delta\omega$, described by

$$K\Delta\omega = -\Delta p^L \tag{2.22}$$

where K is called the droop coefficient and depends on the available amount of spinning reserves (a large increase in generation would bring more reserves to their limits and cause K to drop [25]). Figure 2.4 demonstrates the effect of frequency droop in generation for constant K, where it can be seen that an increase in the disturbance Δp^L, results to a drop in frequency.

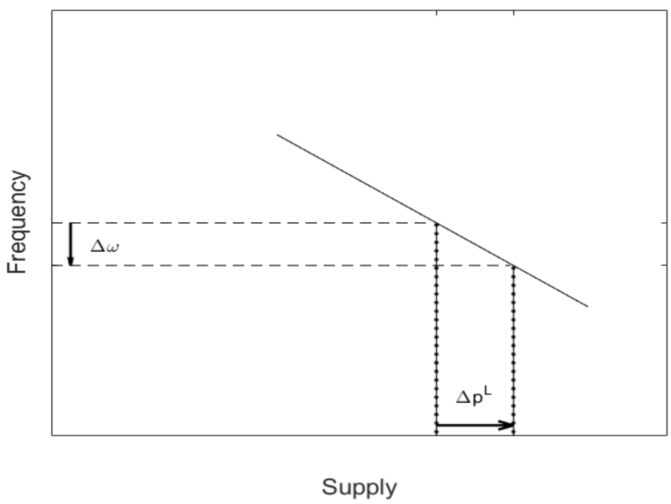

Figure 2.4. Frequency deviation vs. change in power supply as a result of primary frequency control.

It should be noted that system loads also respond to frequency deviations aiding to equalise generation and demand. However, their contribution is limited and frequently neglected.

In brief, primary frequency control action ensures that generation and demand are matched at the expense of some resulting steady state frequency deviation. These effects occur within tens of seconds (hence the name primary control). The resulting frequency deviation is corrected by secondary frequency control action, as we shall see in the following section.

2.3.2 Secondary frequency control in the power grid

The main objective of secondary frequency control is to restore the frequency deviation that results from primary control action ensuring that balance between generation and demand is kept. Traditionally, this is achieved via centralised control action adjusting generation set points and activating available ancillary services. Secondary control action occurs in a timeframe ranging from tens of seconds to about 15 minutes, as a result of the time required to activate spinning reserves and the necessary communication overhead. In many cases, in addition to ensuring frequency return to its nominal value, secondary frequency control is also responsible to ensure that inter-area power transfer agreements are satisfied[5]. Automatic generation control corresponds to a class of typical implementations of secondary control and has been excessively studied in literature [31, 32, 33, 34].

2.3.3 Tertiary control

Tertiary control operates at a slower time scale than secondary control (usually tens of minutes). Its objective is to set the power commands on the generators in a way that minimises the projected cost associated with their operation, taking into account system constrains and making sure that adequate spinning reserves are available for primary and secondary control.

Most commonly, tertiary control is performed by a central operator in each control area. However, in countries where electricity supply systems have been liberalized, the operator does not have direct control over the power plants. In these cases, an actual power market exists and power companies either bid in a central pool or have private contracts to provide energy [35]. The operator's main task in this case is to adjust the bids in order to ensure power constraints satisfaction and that enough reserves for primary and secondary control are available [25].

2.3.4 Optimality in frequency control

As explained in the previous section, economic considerations motivate the selection of generation profiles that minimise projected costs while satisfying physical and operational constrains of the network. The optimal power flow (OPF) problem (e.g. [36, 37]) aims to obtain the generation dispatch that meets the projected demand and

[5]A large interconnected power system is usually divided into a number of interconnected areas which usually follow power transfer agreements with their neighbours. Such agreements can be made between countries and power utility companies.

satisfies operational constrains such as generation and bus voltage limits and power flow constrains. The complexity of the OPF problem makes its implementation in real-time via the primary and secondary timeframes infeasible and consequently no optimality considerations have been typically made within those control schemes. It is therefore important to investigate whether is possible to include any economic considerations within the primary and secondary frequency control schemes such that some simplified form of the OPF problem is taken into account in the control action.

2.4 Demand-side management

Environmental concerns are today an important factor in shaping public opinion and naturally take a high priority in many governments agendas, being a driving factor for increasing investments in renewable sources of energy. However, renewable generation is unpredictable and a large penetration may cause frequent imbalances between generation and demand. This is expected to largely increase the costs of the power network, since fast spinning reserves are generally less efficient. Furthermore, conventional means of generation are slower and hence not able to counterbalance generation and demand. A solution to this problem comes from load side participation which can provide fast response to power imbalances until conventional generation is brought online. Loads ranging from household appliances (such as air-conditioners, refrigerators, and water or space heaters) to large industrial units may adapt their demand accordingly to support the power network for short time periods, causing only a negligible impact in users convenience and comfort levels.

Although the idea of using controllable devices to support the network in power balancing and frequency restoration dates back to the 1970s [38], only recently[6] significant research focus was paid on the topic of demand management [40] both for primary [41, 42] and secondary control [43, 44]. In all these studies, it was shown via simulations that the incorporation of controllable loads within the power network offered significant improvements in system performance.

The introduction of controllable loads will greatly increase the number of contributing elements in the power grid. It is therefore important to study the behaviour of the power network when such heterogeneous dynamics are coupled with the swing

[6]Note also the patent [39] which considers the idea of using frequency as the control signal to control the demand from not critically dependent users, which was named frequency adaptive power-energy re-scheduler (FAPER).

equation and existing frequency control schemes. The study of networks with such a large number of elements shows the importance of designing distributed control schemes that will allow the 'plug and play' operation of loads and at the same time guarantee the stability of the power network. It is an open problem to find the least conservative distributed conditions on the dynamics of generation and controllable demand such that stability is guaranteed on general network topologies.

Another issue raised if controllable loads are to participate in power networks is that of fairness in the power allocation among them. It is important to ensure that the contribution of loads to support the network is equitably distributed. Attempts to resolve this issue in literature involved constructing an appropriate optimisation problem that ensured the desired fairness in the power allocation. Such schemes have been proposed in both primary and secondary control schemes and is evident that a synchronising variable shared among all buses is necessary for optimality. For primary control, frequency has been used as the synchronising variable, allowing for simple decentralised optimality schemes. However, in secondary control, where frequency takes its nominal value, a different synchronising variable has been required and more elaborate control schemes had to be employed usually by making use of exchange of information between neighbouring buses.

2.5 Literature review on frequency control

This section contains a review on the historical development of frequency control schemes and discusses various approaches that have been attempted in literature to address stability and optimality issues in primary and secondary frequency control timeframes. Particular focus is paid in literature where demand side management has been considered. Within this section, we shall often refer to the power network dynamics described in Section 2.2.6, also displayed below for convenience.

$$\dot{\eta}_{ij} = \omega_i - \omega_j, \ (i,j) \in E, \tag{2.23a}$$

$$M_j \dot{\omega}_j = -p_j^L + p_j^M - (d_j^c + d_j^u) - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij}, \ j \in G, \tag{2.23b}$$

$$0 = -p_j^L - (d_j^c + d_j^u) - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij}, \ j \in L, \tag{2.23c}$$

$$p_{ij} = B_{ij} \sin \eta_{ij} - p_{ij}^{nom}, \ (i,j) \in E. \tag{2.23d}$$

2.5.1 Primary frequency control

As discussed in Section 2.3.1, the main objective of primary frequency control is to ensure that generation and demand are balanced. This is achieved by appropriately coupling the system (2.23) with the control dynamics for generation p_i^M and controllable demand d_i^c to ensure generation-demand balance which results to frequency convergence to a constant non-zero value. Note that frequency reaching a steady state suffices to ensure global power balance, as follows by summing equations (2.23b) and (2.23c) at equilibrium. Moreover, it is important to note that at equilibrium all frequencies synchronise to a common value, as follows from (2.23a).

Historical development

The paper of Steinmetz [45] was the first that attempted to look analytically at the problem of frequency stability, where it considered how problematic oscillatory behaviour can be avoided in a model of two coupled synchronous generators. Moreover, during the 1970s several stability investigations where motivated by major stability issues in the North American grid (e.g. Toronto, January 1974 and Missouri/Illinois, February 1978). In [46], a linearised swing equation model was considered and stability guarantees where provided for two classes of distributed control schemes. Furthermore, a thorough discussion on implementable control schemes with analysis based on detailed mathematical frameworks can be found in [26]. Governor droop control is the most common implementation of primary frequency control, where generation output is increased in direct proportion with local frequency deviation from nominal. Detailed models of its implementation have been considered in various studies, such as [25, 26, 47, 48].

Passivity conditions for stability analysis

The notion of passivity (e.g. [49]) has been of great importance in the study of large scale systems and has found many applications within the power system literature. Passivity related approaches with applications on power systems date back to 1980s, where [50] deduces stability guarantees on a system described by the swing equation model (2.23) coupled with first order voltage dynamics. Moreover, the port-Hamiltonian framework has also been used in many recent studies with examples such as [51, 52, 53, 54, 55].

Passivity based approaches have been applied in [56], where a linearised swing equation model that follows from (2.23) was considered along linear frequency damping and no controllable demand, i.e. with $d_i^u = D_i\omega_i$ and $d_i^c = 0$. Furthermore generation dynamics described in the Laplace domain by

$$p_i^M = -K_i(s)\omega_i$$

where included, where $K_i(s)$ is any positive real[7] transfer function. This analysis showed that any passive generation dynamics can be incorporated in the network without affecting its stability properties. Moreover, [57] considered the same network model as [56] but also included first order voltage dynamics in generation, demonstrating asymptotic stability when generation dynamics are described by

$$p_i^M = -K_i^\omega\omega_i - K_i^V(V_i - V_i^{nom})$$

with constants $K_i^\omega, K_i^V > 0$. Hence, passivity is a useful tool to provide stability guarantees on networked systems, such as (2.23). Furthermore, appropriate input-output conditions on the system dynamics can allow for optimality interpretations when the system has reached steady state, as described in the following subsection.

Economic optimality and fairness in primary control

We have seen in Section 2.3.3 that tertiary control is responsible to tune generation setpoints to minimise some measure of cost while taking into account operation constrains. However, traditional implementations do not take into account any optimality considerations within the primary and secondary frequency control timeframes. Hence, the power allocation that results from these shorter timescale schemes fails in general to be optimal allowing particular generators to take an uneven share of the power allocation. Furthermore, fairness in the power allocation becomes particularly important when controllable loads are introduced in the power network to aid in primary and secondary frequency control. Therefore, opportunities exist to improve the current implementation of faster scale frequency control schemes by taking economic considerations into account.

The design of any optimality scheme requires the existence of a nonzero synchro-

[7] The definition of a positive real transfer function can be found in [49, Definition 6.4].

nising variable which enables the appropriate alteration of generation and controllable demand at each bus such that equal marginal costs are attained. In primary frequency control, the presence of the non-zero synchronised frequency conventionally allows it to be used as the synchronising variable without any communication requirements, enabling decentralised optimal control implementations. This topic has been studied in literature, where the steady state values of generation/demand where designed to be solutions of an appropriately constructed optimisation problem which ensured fairness in power allocation as well as balance between generation and demand. An example of using frequency as a synchronising signal is presented in [58], where proportional power allocation is achieved by appropriate selection of generation droop gains. A similar approach was adopted in [59], where again generation droop coefficients where selected proportionally by making use of the local damping coefficients.

In [60], the authors considered a linearisation of the dynamics in (2.23) and assumed constant generation and uncontrollable demand and generation damping described by $d_i^u = D_i \omega_i$. Then, the following optimisation problem was constructed to obtain the optimal allocation among controllable loads

$$\begin{aligned} \underset{\omega, d^c}{\text{minimize}} \quad & \sum_{i \in N} \left(C_{di}(d_i^c) + \tfrac{1}{2} D_i \omega_i^2 \right) \\ \text{subject to} \quad & \sum_{i \in G} p_i^M = \sum_{i \in N} (d_i^c + D_i \omega_i + p_i^L), \\ & d_i^{c,min} \leq d_i^c \leq d_i^{c,max}, \ \forall i \in N. \end{aligned} \tag{2.24}$$

The above problem penalises changes in controllable demand by a strictly convex function C_{di} representing the incurred cost from discutility of loads and those in frequency via a quadratic term. Moreover, the problem includes an equality constrain which ensures balance between generation and demand as well as a constrain on the maximum and minimum deviations in controllable demand which provides a more realistic optimality representation. It was shown that the following decentralised static dynamics[8]

$$d_i^c = \left[(C'_{di})^{-1}(\omega_i) \right]_{d_i^{c,min}}^{d_i^{c,max}} \tag{2.25}$$

ensure convergence to an optimal solution to (2.24). This analysis was among the first to demonstrate how a fair power allocation might be achieved among controllable loads in a decentralised way by considering an appropriate optimisation

[8]Recall that the expression $[q]_a^b$ denotes $\max\{\min\{q, b\}, a\}$.

problem.

The study of [61] extended that of [60] by considering the non-linear swing equation model (2.23) and dynamic generation. To account for the cost incurred from the deviation in generation, an extra term $\sum_{i\in G} C_i(p_i^M)$ was considered in (2.24). Furthermore, the following second order linear model was used to describe generation dynamics.

$$\left.\begin{aligned} \frac{d\alpha_i}{dt} &= -\frac{1}{\tau_{g,i}}\alpha_i + \frac{1}{\tau_{g,i}}p_i^c, \\ \frac{dp_i^M}{dt} &= -\frac{1}{\tau_{b,i}}p_i^M + \frac{1}{\tau_{b,i}}\alpha_i, \end{aligned}\right\} \qquad i \in G. \tag{2.26}$$

It was then shown that the equilibria of the system where optimal when $p_i^c = \left[(C_i')^{-1}(-\omega_i)\right]_{p_i^{c,min}}^{p_i^{c,max}}$. In addition, the following gain constrain was imposed

$$|p_i^c(\omega_i) - p_i^c(\omega_i^*)| \le L_i|\omega_i - \omega_i^*| \tag{2.27}$$

to hold for some constant $L_i < D_i$ in some local neighbourhood of any equilibrium frequency ω_i^*. This condition is not related with the optimality analysis but was imposed in order to provide stability guarantees. In Chapter 4, we show how this condition can be relaxed. Therefore, we have seen that optimality schemes have been studied for both generation and demand so as to achieve the objectives of primary frequency control and at the same time guarantee a fair power allocation.

A similar approach was also employed in [43] on a linearised system which allowed the design of a distributed load control scheme that ensured convergence to a global minimum of (2.24).

2.5.2 Secondary frequency control

Once primary control has stabilised the grid frequency, secondary control is employed, as described in Section 2.3.2, to regulate this frequency back towards its nominal value. The control schemes for generation p_i^M and controllable load d_i^c thus need to be designed so as to drive the solutions of (2.23) to an equilibrium where the frequency takes its nominal value. In contrast to primary control, the fact that the frequency deviations return to zero means that a different variable needs to be used for synchronisation, if an optimal allocation is desired. Therefore, the optimality control schemes involved will typically require some communicated variable. It should be noted that this communication makes it important to carefully consider which devices to include within secondary control as the participation of large numbers of small loads could imply a significant communication overhead. It is therefore likely that provision of secondary control resources might be restricted to generators and larger controllable loads. In that case, the remaining loads (which may have been controllable and hence represented within the d_i^c terms for the purpose of primary control) would be modelled within the uncontrollable terms d_i^u.

Historical development

Studies concerned with frequency regulation through Automatic Generation Control (AGC) date back to the 1950s, with [62, 63] mainly focussing on tie-line based control techniques. Over the following decades, further studies were performed on the topic, mainly considering linearised models of two/multi-area systems [64, 65, 66]. An n-area system was investigated in [64], which considered non-interaction between frequency and tie-line controllers, while in [65] the authors studied a multi-area power network model and gave recommendations to improve stability margins, comparing them with existing regulations of the North American Power Systems Committee. The fact that linear analysis is only justifiable in the presence of small perturbations has been noted in studies considering system nonlinearities such as governor deadband [67] and nonlinear tie-line bias control [68]. Artificial intelligence techniques have been employed to permit the study of models that change according to the operating conditions, thus allowing a far more realistic representation of power systems. For instance, neural network approaches have offered many advantages in the study of systems operating in nonlinear regimes. This approach was applied on a 4-area system with nonlinear turbine dynamics in [69] and on a single-area and a two-area system in [70]. In both cases, there were significant performance improvements compared to integral control action. It should be noted that further to the neural network approach, fuzzy logic [71] and genetic algorithm techniques [72, 73] have also been applied to this problem, all demonstrating satisfactory performance characteristics. Furthermore, [74] developed a switching control scheme where loads adapt their demand to support the network at urgencies and otherwise keep to their nominal operation.

In the early days, the AGC problem was dealt with through centralised control strategies [64, 65, 75]. This approach has the important limitation of requiring communication, computation, and storage infrastructures. Decentralised control techniques appeared later in an effort to deal with these complexities [76, 77, 78]. For further discussion and a more thorough review on AGC consult [79].

Economic optimality and fairness in secondary control

As we have previously discussed in Section 2.3.2, the aim of secondary frequency control is to restore frequency to its nominal value. Traditionally, this task is carried out by generators who adapt their power production by using frequency as a control signal, usually via some integral control action. As mentioned in Section 2.3.4, this results in new generation levels that fail in general to be economically optimal. Furthermore, if controllable devices' participation in secondary control is desired, then it is important to ensure fairness in the power allocation between them. Thus, analogously to the discussion for primary control in Section 2.5.1, these topics present research opportunities to derive control schemes that would ensure fairness and economic optimality in secondary frequency control.

As previously discussed, in order to ensure equality of the users' marginal costs so as to achieve optimality, a nonzero synchronising variable is required. This makes the frequency deviations employed for this purpose in Section 2.5.1 unsuitable to use here, since in secondary control the frequency returns to its nominal value. Therefore, a different variable needs to be synchronised, which presents the need for a communication network to provide the necessary information exchange for that synchronisation to happen.

Several recent studies have attempted to devise control schemes such that the steady state conditions coincide with the solutions of an appropriately constructed optimisation problem which ensures economic optimality and/or fairness in power allocation between loads. An example of such a study is [80], which considered a linearisation of the swing equations (2.23), damping terms $d^u = D_j \omega_j$ with $D_j > 0$, first-order dynamics for generation, and constant demand. The authors then posed the optimisation problem

$$\begin{aligned} \underset{p^M, p}{\text{minimize}} \quad & \sum_{i \in N} C_i(p_i^M) \\ \text{subject to} \quad & \sum_{i \in G} p_i^M = \sum_{i \in G} (p_i^L + \sum_{k: i \to k} p_{ik} - \sum_{k: k \to i} p_{ki}), \end{aligned} \tag{2.28}$$

which penalises deviations from nominal value in power generation via a strictly convex function C_i. If the equality constraint above is satisfied at equilibrium in the dynamics described by (2.23), then $\sum_{i \in N} d_i^u = \sum_{i \in N} D_i \omega_i = \omega \sum_{i \in N} D_i = 0$ holds at steady state, which immediately implies that frequency does indeed return to its nominal value. The authors then introduced auxiliary variables to represent the exchange of information between buses and demonstrated that the closed loop system dynamics resemble a dual form of the optimisation problem (2.28). This equivalence allowed the desired steady state optimisation property to be guaranteed. A similar approach has also been followed in a number of other studies, all including a constraint within the optimisation problem to guarantee that frequency would return to its nominal value at steady state. In [81], for example, a similar system including controllable demand and constant generation was studied. The following optimisation problem, called Frequency Preserving Optimal Load Control (FP-OLC), was constructed penalising the deviation in controllable demand from its nominal value, again via a strictly convex function C_j,

$$\begin{aligned}
&\underset{d^c, d^u, p, R}{\text{minimize}} && \sum_{i \in N} \left(C_i(d_i^c) + \frac{(d_i^u)^2}{2D_i} \right) \\
&\text{subject to} && p_i^M - (d_i^c + d_i^u + p_i^L) = \sum_{k:i\to k} p_{ik} - \sum_{k:k\to i} p_{ki}, \quad i \in N, \\
& && p_i^M - d_i^c - p_i^L = \sum_{k:i\to k} \psi_{ik} - \sum_{k:k\to i} \psi_{ki}, \quad i \in N,
\end{aligned} \tag{2.29}$$

where $\psi_{ij}, (i,j) \in E$ are auxiliary variables that facilitate the design of the problem. The formulation of this optimisation problem ensured that any optimal solution has zero steady state deviation in frequency. In addition, the inclusion within the cost function of a term that is proportional to the frequency deviation, which does not change the solution of the problem, allowed the authors to show via the KKT conditions that for optimality, frequency and the Lagrange multiplier associated with the first constraint needed to be equivalent. Furthermore, the Lagrange variable associated with the second constraint can be conveniently thought of as representing a power command signal. This approach motivated the following intuitive dynamics for this power command signals, which could be used as the requisite exchange variables,

$$\gamma_i \dot{p}_i^c = p_i^M - d_i^c - p_i^L - \sum_{k:i\to k} \psi_{ik} + \sum_{k:k\to i} \psi_{ik}, \ i \in G, \tag{2.30a}$$

$$\gamma_i \dot{p}_i^c = -d_i^c - p_i^L - \sum_{k:i\to k} \psi_{ik} + \sum_{k:k\to i} \psi_{ki}, \ i \in L, \tag{2.30b}$$

$$\gamma_{ik} \dot{\psi}_{ik} = p_i^c - p_k^c, \ (i,k) \in E, \tag{2.30c}$$

where $\gamma_i, \gamma_{ik} > 0$ for all $i \in N, (i,k) \in E$. This optimality scheme[9] is called partial primal dual in [81] since it can be seen to follow from the interpretation of the dual problem of the FP-OLC problem (2.29). It should be noted that the communication graph implicit within (2.30) does not necessarily need to be the same as the graph representing the power network. The optimal controllable demand values are then specified by

$$d_j^c = C_j'^{-1}(\omega_j + p_j^c). \tag{2.31}$$

The dynamics in (2.30c) ensure synchronisation of the power command variables, and therefore enable the closed loop dynamics (2.23),(2.30),(2.31) to achieve an optimal solution of the FP-OLC problem (2.29). This analysis shows that it is possible to design control schemes for controllable loads that will ensure fairness in the power allocation between them.

A similar approach was adopted in [43], using the same dynamics as [81] but adding constraints on power transfers. In [82], the nonlinear swing equations (2.23) were used together with the second-order generation dynamics described in equation (2.26) and static controllable loads, and distributed control schemes where proposed for generation/controllable demand such that the equilibrium points considered were optimal for a prescribed optimisation problem. Stability of the system was guaranteed through the imposition of a gain condition, similar to the one described in equation (2.27). A further study which utilised the nonlinear swing equations and also included voltage dynamics within its analysis is [44]. Furthermore, in [83] the authors imposed steady state conditions that ensure that the power injection alterations in micro-grids are proportional to the users' droop coefficients, guaranteeing fair power allocation. Moreover, [84] demonstrated the solution of an optimisation problem with constraints on transmission and load power consumption, through the use of gradient-based distributed control laws within a nonlinear swing equation model with first-order generation dynamics.

A further approach to obtain optimality guarantees in secondary frequency control, used in [44, 59, 85, 58, 86], is the employment of distributed averaging proportional integral (DAPI) controllers, where the power command dynamics are described by

$$\gamma_i \dot{p}_i^c = -k_i \omega_i - \sum_{k \in \Omega_i} a_{ik}(p_i^c - p_k^c), \ i \in N, \tag{2.32}$$

[9]Note that the same scheme with reversed signs for p_j^M, d_j^c and p_j^L variables is adopted elsewhere in literature with power command being treated as $-p^c$ within this setting. This change does not affect any of the stability and optimality properties and is just a different interpretation of (2.30). In Chapter 6, we study the latter version of this scheme.

where $k_i, \gamma_i > 0$ and $a_{ik} = a_{ki} > 0$ for all $i \in N$ and $(i,k) \in E$ respectively and $\Omega_i = \{k : (i,k) \in E \text{ or } (k,i) \in E\}$, i.e. is the set of buses that are connected with bus i. The dynamics in (2.32) ensure synchronisation of the power command variables which results to frequency taking its nominal value at equilibrium. Furthermore, the power command can be used as a synchronisation variable to solve appropriately constructed optimisation problems, similar to the problem considered in (2.29), which can be extended to include generation costs, when cost functions C_i are quadratic and generation/controllable demand have static dynamics.

Remark 2.2 *Advantages of DAPI controllers compared to the controller described in* (2.30) *lie in their simplicity as they only measure local frequency and exchange a synchronization signal in a distributed fashion without requiring any generation/demand power flow measurements. On the other hand, existing results in this context are limited to the case of proportional active power sharing and quadratic cost functions. The former demonstrates that, when DAPI is considered, the available distributed stability guarantees do not accommodate any first or high order generation and controllable demand dynamics.*

Chapter 3

Mathematical background

Within the thesis, we study the stability and optimality of various classes of dynamical systems. In this chapter, we provide the mathematical background required to follow the analysis within the next chapters, in order to enhance the readability of this manuscript. We consider concepts of stability and optimality that apply to broad classes of dynamics that are of high relevance in the power systems literature. In Chapters 4–6, we consider systems with dynamics described by ordinary differential equations. In this chapter, we describe the solutions of such systems and present applicable theorems that guarantee properties such as existence, uniqueness and continuity with initial conditions. Such properties are essential for those dynamics to represent physical systems. Relevant notions of stability as well as important stability theorems, such as Lyapunov's direct method and Lasalle's theorem are presented and discussed. Moreover, we review discontinuous and hybrid dynamical systems, studied in Chapter 7 of this manuscript. For these types of systems, we define notions of solutions and present relevant stability theorems. Furthermore, a general optimisation problem is considered and conditions for optimality are presented. In all cases, pointers to appropriate references are provided, where detailed proofs, further extensions and intuitive discussion can be found.

This chapter is divided into five subsections. Section 3.1 contains some basic notation and preliminaries. Sections 3.2, 3.3 and 3.4 include definitions of solutions and convergence theorems for systems described by ordinary differential equations, discontinuous dynamics and hybrid dynamics respectively. Finally, optimality conditions are presented in Section 3.5.

The mathematical background relevant to Chapters 4–6 is presented in Sections 3.2 and 3.5 and to Chapter 7 in Sections 3.3 and 3.4.

3.1 Notation and preliminaries

This section describes the basic notation and preliminaries used within the rest of this manuscript.

We use $\mathbb{R}$ and $\mathbb{R}_+$ to denote the sets of real and non-negative real numbers respectively. The set of natural numbers is denoted by $\mathbb{N}$ and the set of natural numbers including zero by $\mathbb{N}_0$. The set of n-dimensional vector with real/natural entries is denoted by $\mathbb{R}^n$/ $\mathbb{N}^n$.

The norm $\|x\|$ of a vector $x \in \mathbb{R}^n$ is a real valued function with the properties: (i) $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$, with $\|x\| = 0$ if and only if $x = 0$, (ii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in \mathbb{R}^n$ and (iii) $\|\alpha x\| = |\alpha| \, \|x\|$, for all $\alpha \in \mathbb{R}, x \in \mathbb{R}^n$.

For a discrete set Σ, let $|\Sigma|$ denote its cardinality. For a set A, let $\bar{A}$ and $\bar{co}(A)$ denote its closure and convex closure respectively. We denote the boundary of a set A by $bndry(A)$. A set is called compact if it is closed and bounded. For a vector field X and a set A, let X_A denote the vector field X restricted on A. For a point $x \in R^n$ and positive constant δ let $B(x, \delta)$ denote the ball of radius δ around x. Moreover, let $\mathcal{B}(\mathbb{R}^n)$ denote the collection of subsets of $\mathbb{R}^n$.

We use the notions of Lebesgue measurable set, zero measure set and Lebesgue measurable function from [87]. Furthermore, for a Lebesgue measurable set A, let $\mu(A)$ denote its Lebesgue measure. For notation convenience, Lebesgue measures will be referred as just measures. When something holds almost everywhere within a measurable set A, it means that it holds everywhere in A except on sets N satisfying $N \subset A : \mu(N) = 0$. For a set A and scalar b, $A \leq b$ denotes that all elements in A are less than or equal to b.

The distance of a point x from a set A is denoted by $dist(x, A)$ and is defined as $\|x - A\| = \inf_{a \in A} \|x - a\|$, for any convenient $\|.\|$.

For a function f we use $dom\ f$ and $rge\ f$ to denote the domain and range of f respectively. The derivative of a function $f(a)$ is denoted by $\frac{df}{da}$ or f'. A function f is said to be of differentiability class C^k if its k-th derivative $f^{(k)}$ exists and is continuous. Furthermore, the expression $f^{-1}(w)$ represents the preimage of the point w under the function f, i.e. $f^{-1}(w) = \{q \colon f(q) = w\}$. When the function f is invertible, f^{-1} then defines the inverse function of f. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite on a neighbourhood D around the origin if $f(0) = 0$ and $f(x) > 0$ for every non-zero $x \in D$. It is positive semidefinite if the inequality > 0 is replaced by ≥ 0. We say that f is positive definite with respect to component x_j if $f(x) = 0$ implies $x_j = 0$, and $f(x) > 0$ for every $x_j \neq 0$. Furthermore, $[q]_a^b$ denotes

$\max\{\min\{q, b\}, a\}$ for $a, b \in \mathbb{R}$, $a \leq b$. The indicator function $\mathbb{1}_S : \mathbb{R}^n \to \{0, 1\}$ of a set $S \subseteq \mathbb{R}^n$ takes the value 1 if its argument belongs to the set S and 0 otherwise. Finally, we denote the derivative of a function $q(t)$ with respect to time by $\dot{q}$ and its Laplace transform by $\hat{q}(s) = \int_0^\infty e^{-st} q(t)\, dt$.

3.2 Ordinary differential equations

In this section we present a formal definition of solutions for ordinary differential equations (ODEs) and consider some of their fundamnetal properties, such as existence, uniqueness and continuous dependence on initial conditions. Such properties are essential for the description of any deterministic physical system and are necessary for the validity of the results within this manuscript. This topic is thoroughly addressed in [49, 88]. We shall first consider the following general form of autonomous or time-invariant ODE systems

$$\frac{dx}{dt}(t) = f(x(t)), \tag{3.1}$$

where $f : D \to \mathbb{R}^n$, $D \subseteq \mathbb{R}^n$, is locally Lipschitz continuous on x following the Definition 3.1 below.

Definition 3.1 *A function $f : X \to Y$ is said to be locally Lipschitz if for every $x \in X$ there exists an open set $U \subseteq X$ containing x and a constant L, called the Lipschitz constant, such that*

$$\|f(x) - f(y)\| \leq L \|x - y\|, \tag{3.2}$$

for all $y \in U$.

Furthermore, if the same L can be chosen for all $x \in X$, f is said to be Lipschitz.

We treat the terms t and x in (3.1) as the time and state variables respectively. We shall refer to f as the vector field and interchangeably use the terms solutions and trajectories to denote time dependent continuous functions $x(t)$ that solve (3.1).

The local existence and uniqueness of solutions of systems described by (3.1) satisfying the continuity property over time and being locally Lipschitz on states, follows from [49, Therorem 3.1], as stated below.

Theorem 3.1 *Consider the system described in* (3.1), *any initial condition* $x(0) = x_0 \in X$. *Then, there exists some* $\delta > 0$ *such that* (3.1) *has a unique solution over* $t \in [0, \delta]$.

Remark 3.1 *Theorem 3.1 establishes that the continuity conditions on* (3.1) *suffice for the local existence and uniqueness of solutions. This result can be extended to global existence and uniqueness if* f *is globally Lipschitz on its second argument (see [49, Theorem 3.2]). However, such condition might be restrictive since it would impose a global growth constrain which is often violated in many practical examples. In contrast, the local Lipschitz condition is essentially a smoothness condition and hence much more practical. For these reasons we mainly use a local Lipschitz condition for the study of ODE's within this manuscript.*

Further to the existence and uniqueness of solutions, a desirable property of systems described by (3.1) is the continuous dependence on initial conditions, demonstrated by the following theorem which follows from [49, Theorem 3.4].

Theorem 3.2 *Consider solutions* $y(t)$ *and* $z(t)$ *of* (3.1) *defined within some time interval* $[0, T]$ *with initial conditions* $y(0) = y_0$ *and* $z(0) = z_0$ *respectively. Then, there exists a constant* L *such that*

$$\|y(t) - z(t)\| \leq \|y_0 - z_0\| \, e^{Lt}, \tag{3.3}$$

for all $t \in [0, T]$.

Remark 3.2 *The two theorems presented above ensure that systems with dynamics described by* (3.1) *are locally well-defined and well-behaved. Hence, these results show that* (3.1) *can realistically describe physical systems, justifying its use within the rest of this manuscript.*

Stability - Convergence

In this subsection, we study the stability of equilibria of systems described by (3.1). We provide a mathematically robust definition of stability and present two methods, Lyapunov direct method and Lasalle's theorem, that are useful to show when an equilibrium point is stable.

Before proceeding with notions and results on stability, we need to first define what an equilibrium is [49].

Definition 3.2 *A state $\hat{x} \in \mathbb{R}^n$ is called an equilibrium point of* (3.1) *if $f(\hat{x}) = 0$.*

Remark 3.3 *A point $\hat{x}$ is said to be an equilibrium if it has the property that whenever the state of the system starts at $\hat{x}$, it remains at $\hat{x}$ for all future times.*

The definitions of stable, asymptotically stable and unstable equilibria are provided below [49].

Definition 3.3 *An equilibrium $\hat{x}$ of* (3.1) *is called stable if for all $\epsilon > 0$ exists $\delta > 0$ such that $\|x_0 - \hat{x}\| < \delta$ implies that $\|x(t) - \hat{x}\| < \epsilon$ for all $t \geq 0$. Otherwise, the equilibrium is said to be unstable.*

Definition 3.4 *An equilibrium of* (3.1) *is called locally asymptotically stable if it is stable and exists $\delta > 0$ such that $\|x_0 - \hat{x}\| < \delta$ implies that $\lim_{t\to\infty} x(t) = \hat{x}$. If this holds for any δ, then the equilibrium is called globally asymptotically stable.*

Remark 3.4 *An equilibrium is called stable if when a solution starts close to it, then it always remains close to it. If any solution that starts close to an equilibrium converges to it as time tends to infinity, then this equilibrium is called asymptotically stable. Furthermore, if convergence to some equilibrium occurs from any initial condition, then this equilibrium is called globally asymptotically stable.*

There are several ways to show equilibrium stability depending on the nature of the system, such as the Nyquist criterion [89] or the linearisation method [49]. Below, we present two useful approaches to show stability, the Lyapunov direct method and Lasalle's theorem, both proven in [49, Chapter 4].

A Lyapunov function is strongly related to the energy of a dissipative system and that's why is often referred as 'energy function' [49]. For a system described by (3.1) with an equilibrium at $x = 0$ (which can be assumed without loss of generality, see [49]), the Lyapunov direct method is as follows:

Theorem 3.3 *Assume that $x = 0$ is an equilibrium point of* (3.1) *and let $V : S \to \mathbb{R}$ be a continuously differentiable function defined in some open region $S \subset \mathbb{R}$ containing the origin, such that*

1. $V(0) = 0$,

2. $V(x) > 0, \forall x \in S - \{0\}$,

3. $\dot{V}(x) \leq 0$, $\forall x \in S$.

Then $x = 0$ is a stable equilibrium of (3.1). Moreover, if $\dot{V}(x) < 0, \forall x \in S - \{0\}$, then the equilibrium point is asymptotically stable.

Note that for stable linear systems, there always exists some quadratic function that satisfies the above criteria. Moreover, a function that satisfies the conditions of Theorem 3.3 along the trajectories of (3.1) is called a Lyapunov function of (3.1).

A useful result, applicable to autonomous systems described by (3.1), that complements Theorem 3.3 is Lasalle's theorem or Lasalle's invariance principle. It has been developed by the independent studies of [90, 91] and provides conditions that guarantee convergence of solutions to a quantifiable invariant set. Below, we define an invariant set and state Lasalle's invariance principle [49, 91].

Definition 3.5 *A set of states $S \subseteq \mathbb{R}^n$ of (3.1) is called positively invariant if for all $x(t_0) = x_0 \in S$ and all $t \geq t_0 \geq 0, x(t) \in S$.*

Theorem 3.4 *Let $\Omega \subset S$ be a compact set that is positively invariant with respect to (3.1). Let $V : S \to \mathbb{R}$ be a continuously differentiable function such that*

$$\dot{V}(x) \leq 0, \quad \forall x \in \Omega. \tag{3.4}$$

Let E be the set of all points in Ω where $\dot{V}(x) = 0$ and M the largest invariant set in E. Then all trajectories starting in Ω approach M as $t \to \infty$.

Remark 3.5 *Lasalle's theorem states that if a Lyapunov function exists within some invariant set S, then any solution that initiates within S will converge to the largest invariant set M where the Lyapunov function's time derivative is zero. As a special case of the theorem, if the set M contains only one point, then that point is a locally asymptotically stable equilibrium. Lasalle's invariance principle is very useful as it allows the deduction of stability guarantees for systems where a number of states converge while the rest are allowed to oscillate.*

3.3 Discontinuous dynamical systems

In this section we review some basic concepts of solutions and convergence for systems with discontinuous dynamics, studied in Chapter 7. We define Filippov solutions, which are of particular interest in this setting, and provide results that guarantee their existence and uniqueness. Moreover, we present a relevant invariance principle used in the stability analysis of Chapter 7.

Consider a system with state $x \in \mathbb{R}^n$ and dynamics described by

$$\dot{x}(t) = X(x(t)), \tag{3.5}$$

where $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is allowed to be discontinuous. The potential presence of discontinuities does not fulfil the requirements of Theorem 3.1 and therefore the existence of classical solutions is not guaranteed within this setting. This has been demonstrated in literature (e.g. [92]), with realistic examples of systems with discontinuous dynamics where classical solutions do not exist. Hence, a different notion of solution needs to be employed to study the behaviour of such systems.

Filippov solutions (see e.g. [92, 93]) are a convenient alternative to classical solutions, widely used in the study of discontinuous systems described by (3.5). They allow the study of exotic behaviours, such as the case of an infinite amount of transitions[1] within some finite time, a phenomenon known as Zeno behaviour.

The notion of a Filippov set valued map is essential to define Filippov solutions. For any, potentially discontinuous, function $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the Filippov set valued map $F[X] : \mathbb{R}^n \rightarrow \mathcal{B}(\mathbb{R}^n)$ is defined as,

$$F[X](x) \equiv \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \bar{co}(X(B(x,\delta) \setminus S)), x \in \mathbb{R}^n, \tag{3.6}$$

where $\bigcap_{\mu(S)=0}$ denotes the intersection over all sets S of Lebesgue measure zero. The dynamical system below, which is a differential inclusion, is often considered to study the behaviour of systems described by (3.5),

$$\dot{x}(t) \in F[X](x(t)). \tag{3.7}$$

Remark 3.6 *Systems* (3.5) *and* (3.7) *coincide when X is continuous since at this occasion the set valued map will contain only a single point. The setting in* (3.7) *aids in the definition of Filippov solutions, presented below, that can describe system behaviour in more general settings where classical solutions do not exist.*

Filippov solutions for systems described by (3.7) are defined below.

Definition 3.6 *For systems described by* (3.7)*, a Filippov solution is defined as an absolutely continuous map $x : [0, t_1] \rightarrow \mathbb{R}^n$ that satisfies* (3.7) *for almost all $t \in [0, t_1]$. Furthermore, a Filippov solution is called maximal if it cannot be extended forward in time.*

[1]By transition, we mean a discontinuous change in the vector field.

Remark 3.7 *The definition of Filippov solutions is based on set valued maps, where instead of looking at the value of the vector field at each particular point, we are also interested in the neighbouring values. Specifically, for $x \in \mathbb{R}^n$, the vector field X is evaluated at all points belonging to $B(x, \delta)$, which is the open ball of radius $\delta > 0$, centered at x. Furthermore, an arbitrary set of measure zero in $B(x, \delta)$ is excluded when evaluating X, so that the outcome is the same for two vector fields that differ on a zero measure set. Consequently, two vector fields that differ on zero measure sets will have the same Filippov solutions. Closely related notions of solutions are Krasovskii [94] and Sentis [95] solutions.*

A useful theorem that guarantees the existence and uniqueness of Filippov solutions of (3.7) is presented below [96]. For further study, a thorough investigation on the topic is presented in [92].

Theorem 3.5 *Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a piesewise continuous vector field, with $\mathbb{R}^n = D_1 \cup D_2$. Let $S_X = bndry(D_1) = bndry(D_2)$ be the set of points at which X is discontinuous, and assume that S_X is a C^2-manifold. Furthermore, assume that, for $i \in \{1, 2\}$, $X_{\bar{D}_i}$ is*[2] *continuously differentiable on D_i and $X_{\bar{D}_1} - X_{\bar{D}_2}$ is continuously differentiable on S_X. If, for each $x \in S_X$, either $X_{\bar{D}_1}(x)$ points into D_2 or $X_{\bar{D}_2}(x)$ points into D_1, then there exists a unique Filippov solution of* (3.5) *starting from each initial condition.*

Remark 3.8 *Theorem 3.5 provides sufficient conditions for the existence and uniqueness of Filippov solutions. Note that the piecewise continuity hypothesis on X within the sets D_1 and D_2 guarantees the uniqueness of solutions within each one of them. The additional conditions guarantee that uniqueness is not dirsupted by the discontinuities. The main requirement of the theorem is that the vector field points towards a discontinuity from at least one side of its boundary. Otherwise, if the vector field around a discontinuity points away from it from both sides, it is intuitive that any solution starting (or passing) from that discontinuity will be non-unique.*

Before stating an important invariance principle for discontinuous systems, used to show convergence of Filippov solutions in Chapter 7, we need to first define the notions of equilibrium and weakly invariant set within this context.

Definition 3.7 *A state $\hat{x}$ is called an equilibrium point of* (3.7) *if $0 \in F[X](\hat{x})$.*

[2]It should be clear that the term $X_{\bar{D}_i}$ denotes the vector field X restricted on $\bar{D}_i$.

Definition 3.8 *A set Ω is said to be a weakly invariant set for* (3.7) *if through each point $x_0 \in \Omega$ there exists a maximal solution of* (3.7) *lying in Ω.*

The following invariance principle follows from [97, Theorem 3]. Within it, let S_{x_0} be the set of solutions of (3.7), denoted by $\phi(t)$, initiating at x_0. The definition of a regular function is quite common in literature (e.g. [97, p.363], [98, p.39]) but is omitted here for brevity. Within the proof, we use the following notion to describe the set valued derivative of a function V,

$$\dot{\tilde{V}}(x) = \{a \in \mathbb{R} : \exists v \in F[X(x)] \text{ such that } V'(x)^T v = a\}. \tag{3.8}$$

Note that the invariance principle presented in [97] is a slightly more general Theorem than Theorem 3.6, permitting Lyapunov functions that are not continuously differentiable. However, this generality is redundant within the context of the results presented in this manuscript. Within Theorem 3.6 we use the following definition for a Lyapunov function.

Definition 3.9 *A Lyapunov function for* (3.7) *is a positive definite, continuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that for each solution $\phi(.)$ of* (3.7) *on $I \subseteq \mathbb{R}$ and for all $t_1, t_2 \in I$*

$$t_1 \leq t_2 \Rightarrow V(\phi(t_2)) \leq V(\phi(t_1)).$$

Theorem 3.6 *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous and regular Lyapunov function for* (3.7)*. Assume that for some $l > 0$, the connected component L_l of the level set $\{x \in \mathbb{R}^n : V(x) \leq l\}$ such that $0 \in L_l$ is bounded. Let $x_0 \in L_l$ and solutions $\phi(.) \in S_{x_0}$. Furthermore, let $Z_V = \{x \in \mathbb{R}^n : 0 \in \dot{\tilde{V}}(x)\}$ and M be the largest weakly invariant subset of $\bar{Z}_V \cap L_l$. Then $dist(\phi(t), M) \to 0$ as $t \to +\infty$.*

Remark 3.9 *Theorem 3.6 is a generalisation of Lasalle's invariance principle to systems with discontinuous vector fields*[3]*. It allows to deduce convergence of solutions to the largest weakly invariant set where $0 \in \dot{\tilde{V}}(x)$. The fact that convergence is deduced for a weakly invariant set (i.e. not all solutions but at least one remains in the set) highlights the importance of having unique solutions to use the above theorem in practical applications.*

[3] For related studies, also see [99, 100].

3.4 Hybrid dynamical systems

A combination of continuous and discrete dynamics captures a rich dynamical behaviour that cannot be encountered in purely discrete or continuous systems. Such systems are called hybrid. Their study is of particular interest since it includes a broad range of relevant and realistic systems but at the same time introduces additional challenges. In this section we go through some basic concepts of hybrid systems, used later within Chapter 7 of this manuscript, in order to enhance the readability of this thesis. Moreover, we present a relevant invariance principle that is used for the stability analysis of Chapter 7.

A hybrid dynamical system consists of two maps describing continuous and discrete time dynamics and corresponding regions on which these dynamics apply. In particular, we consider systems with state $z \in \Lambda \subseteq \mathbb{R}^n$ and dynamics described by

$$\dot{z} = f(z), z \in C, \tag{3.9a}$$

$$z^+ = g(z), z \in D, \tag{3.9b}$$

where $C, D \subset \Lambda$ are closed sets, $f(z) : C \to E \subset \Lambda$ and $g(z) : D \to F \subset \Lambda$ are outer semicontinuous[4], non-empty maps describing respectively the continuous and discrete behaviour of the system. Note that $z^+ = g(z)$, where $z^+ = z(t^+)$ and $t^+ = \lim_{\epsilon\to 0} t + \epsilon$, represents a discrete dynamical system where z^+ is determined by the current value of the state z and the update rule given by g. Moreover, f is assumed to locally bounded, i.e. for every compact $K \subset \Lambda$ there exists a compact $K' \subset \mathbb{R}^n$ such that $f(K) \subset K'$. Note that Λ represents the state space where the hybrid variable z evolves.

To study the behaviour of hybrid systems described by (3.9), we need to define the notions of equilibrium, hybrid time domain and hybrid solutions. The definitions are borrowed from [102, 103].

Definition 3.10 *A point $z^* = (x^*, \sigma^*)$ is called an equilibrium of (3.9) if $f(z^*) = 0, z^* \in C$ or $z^* = g(z^*), z^* \in D$.*

Definition 3.11 *A hybrid time domain is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}_0$ consisting of, potentially infinite, time intervals of the form $[t_\ell, t_{\ell+1}] \times \{\ell\}$, where $0 = t_0 \leq t_1 \leq \dots$, or of finitely many such intervals with the last one possibly of the form $[t_\ell, t_{\ell+1}] \times \{\ell\}$,*

[4]A set valued mapping $f : O \to \mathbb{R}^n$ is outer semicontinuous if for every convergent sequence of x_i, $\lim x_i \in O$ and every convergent sequence of $\zeta_i \in f(x_i)$, $\lim \zeta_i \in f(\lim x_i)$ [101].

$[t_\ell, t_{\ell+1}) \times \{\ell\}$ or $[t_\ell, \infty) \times \{\ell\}$. Consider a function $z(t, \ell)$ defined on a hybrid time domain K such that for every fixed ℓ, $t \to z(t, \ell)$ is a locally absolutely continuous function on the interval $T_\ell = \{t : (t, \ell) \in K\}$. The function $z(t, \ell)$ is a solution to (3.9) if $z(0, 0) \in \Lambda_0$ and for each ℓ it holds that

$$
\begin{aligned}
&\dot{z}(t, \ell) = f(z(t, \ell)), \text{ for almost all } t \in T_\ell, \\
&z(t, \ell) \in C, \text{ for all } t \in [\min T_\ell, \sup T_\ell), \\
&z(t, \ell + 1) = g(z(t, \ell)), \; z(t, \ell) \in D \\
&\text{for all } (t, \ell) \in K \text{ such that } (t, \ell + 1) \in K.
\end{aligned}
$$

A solution $z(t, \ell)$ is complete if K is unbounded. A solution is maximal if it cannot be extended[5]*.*

Remark 3.10 *A hybrid time domain consists of a time variable t and a jump variable j. Hence, for $(t, j) \in dom z$, $z(t, j)$ represents the state of the hybrid system after time t and j jumps. Every hybrid solution z has a hybrid time domain associated with it. However, for a given hybrid system, not every hybrid time domain is a domain of some solution to this system. This phenomenon extends further than the case where solutions blow up in finite time and hence their time domains are only defined in a bounded subset of $[0, \infty]$. For example, when a hybrid system cannot admit more than one consecutive jump (e.g. when function g maps from D to some space F such that $F \cap D = \emptyset$), then a hybrid time domain that contains $(t, j), (t, j+1)$ and $(t, j+2)$ for any $t \in \mathbb{R}_+, j \in \mathbb{N}_0$ cannot be in the domain of any solution. Hence, the time domain complements the solution of a hybrid system and must be generated along with it rather than picked with the hybrid system.*

The existence and uniqueness of solutions to systems described by (3.9) have been well studied in literature. The interested reader is referred to [102] for a thorough investigation of the topic.

Below we provide the definition of a weakly invariant set for hybrid systems and present an important invariance result that follows from [101], which is used in the convergence analysis of Chapter 7. We use S_{z_0} to denote the subset of hybrid trajectories z in domain S starting at z_0 and S_H to denote the set of all solutions to (3.9). A trajectory $z \in S$ is called precompact if it is complete and $\overline{rge\ z} \subset \Lambda$, where Λ is the state space of (3.9).

[5]That is, there is no other solution $\tilde{z}$ with time domain $\tilde{K}$ such that K is a proper subset of $\tilde{K}$ and $\tilde{z}$ agrees with z on K.

Definition 3.12 *For the set of hybrid trajectories S and hybrid domain K, the set $M \subset \Lambda$ is said to be:*

a) weakly forward invariant (with respect to S) if for each $z_0 \in M$, there exists at least one complete hybrid trajectory $z \in S_{z_0}$ with $z(t,j) \in M$ for all $(t,j) \in K$.

b) weakly backward invariant (with respect to S) if for each $q \in M, N > 0$, there exists $z_0 \in M$ and at least one hybrid trajectory $z \in S_{z_0}$ such that for some $(t^*, j^*) \in K, t^* + j^* \geq N$, we have $z(t^*, j^*) = q$ and $z(t,j) \in M$ for all $(t,j) \leq (t^*, j^*), (t,j) \in K$.

c) weakly invariant (with respect to S) if it is both weakly forward invariant and weakly backward invariant.

We now state an invariance principle, Theorem 3.7, for systems described by (3.9), that is used in the setting studied in this thesis. Note that within Theorem 3.7, we make use of the terms in equation (3.11) below.

Theorem 3.7 *Given a hybrid system described by* (3.9)*, let $V : \Lambda \to \mathbb{R}$ be continuous on Λ and locally Lipschitz on a neighbourhood of C. Suppose that $U \subset \Lambda$ is nonempty, and that $x \in S_H$ is precompact with $\overline{rge\, x} \subset U$. If*

$$u_C(z) \leq 0, u_D(z) \leq 0,$$

for all $z \in U$, then for some constant $r \in V(U)$, x approaches the largest weakly invariant set in

$$V^{-1}(r) \cap U \cap [u_C^{-1}(0) \cup (u_D^{-1}(0) \cap g(u_D^{-1}(0)))]. \tag{3.10}$$

Within Theorem 3.7, the following notation is used:

$$u_C(x) = \begin{cases} \max_{v \in f(x)} \max_{\zeta \in V'(x)} \langle \zeta, v \rangle, x \in C, \\ -\infty, \text{ otherwise,} \end{cases} \tag{3.11a}$$

$$u_D(x) = \begin{cases} \max_{\zeta \in g(x)} \{V(\zeta) - V(x)\}, x \in D, \\ -\infty, \text{ otherwise,} \end{cases} \tag{3.11b}$$

$$f^{-1}(r) = \{x \in dom f : f(x) = r\}. \tag{3.11c}$$

Remark 3.11 *Theorem 3.7 is a means to deduce convergence of hybrid solutions to a weakly invariant set. The fact that guarantees are provided only for a weakly invariant set (i.e. a solution exists where the set is invariant, hence not all solutions necessarily remain within the set) shows the importance of deducing uniqueness of solutions in the setting. Such properties are shown to hold in Chapter 7 where the above theorem is applied. Moreover, it is shown in [101] that when the solutions do not exhibit Zeno behaviour, then* (3.10) *can be simplified to* $V^{-1}(r) \cap U \cap u_C^{-1}(0)$.

3.5 Optimality

We aim to create schemes that not only guarantee convergence to some equilibrium point but also esnure that the equilibrium is optimal in some aspect. To achieve this, we use tools from optimisation literature that allow the characterisation of an equilibrium point in terms of optimality. In this section, we present an important tool that allows the deduction of optimality guarantees, namely the Karush - Kuhn - Tucker (KKT) conditions [104]. The usefulness of the KKT conditions follows from their generality which allows them to be applied to a broad variety of problems..

A general form of a cost minimisation problem with inequality and equality constrains is presented below

$$\min_x f(x) \tag{3.12}$$

subject to

$$g_i(x) \leq 0, \ i = 1, \dots, m, \tag{3.13a}$$

$$h_i(x) = 0, \ i = 1, \dots, l. \tag{3.13b}$$

where $x \in \mathbb{R}^n$ is the decision variable, $f : \mathbb{R}^n \to \mathbb{R}$ the cost function and $g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \dots, m$, and $h_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \dots, l$, functions related to the problem constrains.

It is assumed that functions $f, g_1, \dots, g_m$, and $h_1, \dots, h_l$ are all continuously differentiable. This is the case in the optimisation problems considered in Chapters 4 and 6. We then show how this assumption might be relaxed, presenting optimality conditions that extend to functions that are not continuously differentiable, as is the case considered in Chapter 5.

The derivation of the KKT conditions follows by considering the Lagrangian and the primal and dual problems of (3.12)–(3.13), as thoroughly described in [104, Chapter 5]. They provide necessary and sufficient optimality conditions for the problem (3.12)–(3.13), when the cost function and inequality constrains are convex and the equality constrains affine. It should be noted that $\mu_i, i = 1, \ldots, m$ and $\lambda_j, j = 1, \ldots, l$ within the theorem are auxiliary variables (called Lagrange multipliers) that relate with the inequality and equality constrains in (3.13).

The conditions state that if $f, g_i, i = 1, \ldots, m$, are convex and $h_i = i = 1, \ldots, l$ affine continuously differentiable functions, then (x^*, λ^*, μ^*) is an optimal solution of (3.12)–(3.13) if and only if the equations (3.14) below hold.

$$\nabla f(x^*) + \sum_{i=1}^{m} \mu_i^* \nabla g_i(x^*) + \sum_{j=1}^{l} \lambda_j^* \nabla h_j(x^*) = 0, \tag{3.14a}$$

$$\mu_i^* \, g_i(x^*) = 0, \ \forall i = 1, \ldots, m, \tag{3.14b}$$

$$g_i(x^*) \leq 0, \ i = 1, \ldots, m, \tag{3.14c}$$

$$h_i(x^*) = 0, \ i = 1, \ldots, l, \tag{3.14d}$$

$$\mu_i^* \geq 0, \ i = 1, \ldots, m. \tag{3.14e}$$

Remark 3.12 *The KKT conditions provide necessary and sufficient conditions for optimality when a convex optimisation problem with differentiable objective and constraint functions satisfies Slater's condition (i.e. some feasible solution exists). The KKT conditions are very important in optimisation literature with many algorithms for convex optimisation being able to be interpreted as methods for solving them.*

The importance of the KKT conditions lies in their applicability to a broad range of problems. However, as we shall see in Chapter 5, the requirement that cost and constraint functions are continuously differentiable can be restrictive in some cases, making the conditions not applicable on important classes of dynamical schemes. Below, we discuss how the requirement of continuous differentiabiity of the cost and constraint functions can be relaxed by making use of subgradient techniques. The notions of subgradient and subdifferential (e.g. [105]) are defined below.

Definition 3.13 *Given a convex function $f : I \to \mathbb{R}$, a subgradient of f at a point $x \in I \subseteq \mathbb{R}^n$ is any $v \in \mathbb{R}^n$ such that $f(y) - f(x) \geq v^T(y - x)$ for all $y \in I$. The set* *of all subgradients of f at x is called the subdifferential of f at x and is denoted by $\partial f(x)$.*

Using this notion, the KKT conditions can be relaxed, allowing to consider cost and constraint functions that are not necessarily continuously differentiable.

The relaxed conditions state that if $f, g_i, i = 1, \ldots, m$, are convex and $h_i = i = 1, \ldots, l$ affine continuous functions, then (x^*, λ^*, μ^*) is an optimal solution of (3.12)–(3.13) if and only if (3.15) hold.

$$0 \in \partial f(x^*) + \sum_{i=1}^{m} \mu_i^* \partial g_i(x^*) + \sum_{j=1}^{l} \lambda_j^* \partial h_j(x^*), \tag{3.15a}$$

$$\mu_i^* \, g_i(x^*) = 0, \ \forall i = 1, \ldots, m, \tag{3.15b}$$

$$g_i(x^*) \leq 0, \ i = 1, \ldots, m, \tag{3.15c}$$

$$h_i(x^*) = 0, \ i = 1, \ldots, l, \tag{3.15d}$$

$$\mu_i^* \geq 0, \ i = 1, \ldots, m. \tag{3.15e}$$

Remark 3.13 *The subgradient KKT conditions differ from the original KKT conditions in condition* (3.15a) *where the potential non-differentiability of the functions is taken into account by making use of the subgradient of a function, as defined in Definition 3.13. The subgradient KKT conditions are applied in the optimality analysis presented in Chapter 5.*

Part I

Primary frequency regulation with load-side participation

Synopsis of contribution

This part focuses in the study of power network behaviour when controllable loads are considered within the primary frequency control time-frame. The inevitably highly distributed nature of loads shows the need for distributed control schemes, introducing additional complexity in their analysis and design requirements. The main contributions of this part are outlined below.

In Chapter 4 we introduce a passivity framework that allows the design of decentralised generation and demand control schemes for primary frequency regulation such that stability and fairness in allocation are guaranteed. The main stability condition in this chapter, Assumption 4.2, imposes a decentralised passivity requirement on the aggregate dynamics of generation and demand at each bus. Theorem 4.1 demonstrates convergence of solutions for the considered system. Furthermore, conditions that guarantee the optimality of an appropriately constructed problem are provided, as demonstrated in Theorem 4.2. The analysis in this chapter allows for relaxed stability conditions for both linear and non-linear systems. Furthermore, we present various classes of dynamics used in recent studies that fit within the proposed framework, along with practically relevant cases that have not been considered in literature within this context.

Chapter 5 considers the problem of designing distributed generation and demand schemes to provide ancillary service in the primary frequency control timeframe. In such schemes, it is desired that loads respond to frequency deviation only when a particular threshold is exceeded, providing ancillary services at urgencies and to otherwise keep to their nominal operation. This leads to the study of non-linear schemes with discontinous derivatives, which imposes additional complexity in the optimality interpretation, resolved by employing subgradient methods. The main result in this chapter is presented in Theorem 5.2 which provides optimality conditions for the considered systems. Furthermore, convergence is deduced by applying the passivity based arguments presented in Chapter 4. Finally, small gain results tailored for the dynamics considered in this chapter are presented.

Our analytic results are verified in both chapters with realistic simulations on well accepted benchmarks.

This work was taken in collaboration with Eoin Devane, under the supervision of Dr. Ioannis Lestas. Eoin's contribution has been valuable with insightful comments and help in formalising the problems and proofs.

Chapter 4

Primary frequency regulation with load-side participation: stability and optimality

In this chapter, we present a method to design distributed generation and demand control schemes for primary frequency regulation in power networks that guarantee asymptotic stability and ensure fairness of allocation. We impose a passivity condition on net power supply variables and provide explicit steady state conditions on a general class of generation and demand control dynamics that ensure convergence of solutions to equilibria that solve an appropriately constructed network optimization problem. We also show that the inclusion of controllable demand results to a drop in steady state frequency deviations. We discuss how various classes of dynamics used in recent studies fit within our framework and show that this allows for less conservative stability and optimality conditions. We illustrate our results with simulations on the IEEE 68-bus transmission system and the IEEE 37-bus distribution system with static and dynamic demand response schemes.

4.1 Introduction

We have discussed in Chapter 2 how the large scale integration of renewable sources of energy within the power grid is expected to cause fast changes in generation, making power imbalances increasingly frequent due to the inability of conventional means of generation to counter-balance them [106, 107]. Load participation is considered to be one potential solution to this problem, providing fast response to power changes. Household appliances like air conditioning units, heaters, and refrigerators can be controlled to adjust frequency and regulate power imbalances. Although the idea dates back to the 1970s [38], research attention has recently increasingly focused on the concept of controllable demand [108, 109, 110] with particular consideration given to its use for primary control [41, 42].

An issue of fairness and optimality in the allocation is, however, raised if highly distributed schemes are to be used for frequency control at faster timescales, such as schemes involving controllable loads. As discussed in Section 2.5.1, recent studies have attempted to address this issue by devising control schemes which solve an optimization problem guaranteeing a fair allocation between them. This approach has been studied for primary and also for secondary control. We consider here primary rather than secondary control in order to avoid the additional communication that would be necessary to get a fair allocation if controllable demand were used in the latter case. This is because it is evident that a synchronizing variable is necessary to achieve optimality, allowing all nodes to adapt their generation and controllable demand so as to attain equal marginal costs. In primary control, frequency deviation from the nominal value can be used for this purpose, allowing decentralized control to be achieved [60, 61, 111].

In this chapter, we consider the network model derived in Chapter 2, described by nonlinear swing equations. We consider a general class of dynamics for power generation and controllable demand, on which we impose appropriate conditions so as to achieve stability of the equilibrium points and an optimization interpretation of those. This allows us to guarantee, for a wide variety of possible generation and demand dynamics, convergence to a power allocation that solves an appropriately constructed optimization problem, thus ensuring fairness in this allocation. The class of dynamics considered incorporates control schemes using only local frequency measurements as input signals, and we demonstrate that this is sufficient to enable them to take the right decisions so as to converge to a global optimum, thus allowing distributed control. We illustrate the applicability of our approach by demonstrating that various dynamics that have been used in recent interesting studies, such as [60] and [61], can be incorporated within our framework, and we show that the analysis presented in the chapter can give less conservative stability and optimality conditions.

It should be noted that one of the distinctive features of our analysis is that optimality of the power allocation is provided via appropriate conditions on the input/output properties of the systems considered.

The chapter is organized as follows. Sections 4.2 and 4.3 give some basic notation and preliminaries. In Section 4.4, we present the power network model and section 4.5 presents our main results, which are proved in Appendix A. In Section 4.6, we discuss how our analysis relates to other important studies. Section 4.7 illustrates our results through simulations on the IEEE 68-bus transmission system and the IEEE 37-bus distribution system. Finally, conclusions are drawn in Section 4.8.

4.2 Notation

Within this chapter we shall make use of the notation presented in Section 3.1.

For a system as in (4.1) where $x = \bar{x}$, $u = y = 0$ is an equilibrium point, the $\mathcal{L}_2$-gain is defined as $\sup_{\|u\|_2 \neq 0} \frac{\|y\|_2}{\|u\|_2}$ with $x(0) = \bar{x}$, where the $\mathcal{L}_2$-norm is $\|f\|_2 = \sqrt{\int_0^\infty f^2(t)\, dt}$. It can be shown that for a stable linear system with transfer function $G(s)$ its $\mathcal{L}_2$-gain is given by $\sup_{\phi \in \mathbb{R}} |G(j\phi)|$ [112].

4.3 Preliminaries

Throughout the chapter we will consider dynamical systems describing generation and demand with input $u(t) \in \mathbb{R}$, state $x(t) \in \mathbb{R}^m$, and output $y(t) \in \mathbb{R}$ with a state space realization of the form

$$\begin{aligned} \dot{x} &= f(x, u), \\ y &= g(x, u), \end{aligned} \tag{4.1}$$

where $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ and $g : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous. We assume in system (4.1) that given any constant input $u(t) \equiv \bar{u} \in \mathbb{R}$, there exists a unique locally asymptotically stable equilibrium point $\bar{x} \in \mathbb{R}^m$, i.e. $f(\bar{x}, \bar{u}) = 0$. The region of attraction[1] of $\bar{x}$ is denoted by X_0. We also define the static input-state characteristic map $k_x : \mathbb{R} \to \mathbb{R}^m$,

$$k_x(\bar{u}) := \bar{x}.$$

Based on this, we can also define the static input-output characteristic map $k_y : \mathbb{R} \to \mathbb{R}$,

$$k_y(\bar{u}) := g(k_x(\bar{u}), \bar{u}). \tag{4.2}$$

[1]That is, for the constant input $u = \bar{u}$, any solution $x(t)$ of (4.1) with initial condition $x(0) \in X_0$ must satisfy $x(t) \to \bar{x}$ as $t \to \infty$. The definition of local asymptotic stability also implies that X_0 has nonempty interior.

The requirement that for each constant input to (4.1) there exists a unique equilibrium point could be relaxed to require only isolated equilibria, however, we assume it here to simplify the presentation.

4.4 Problem formulation

4.4.1 Network model

The power network model considered follows from the analysis presented in Section 2.2.6 and is described by a graph (N, E) where $N = \{1, 2, \dots, |N|\}$ is the set of buses and $E \subseteq N \times N$ is the set of transmission lines connecting the buses. There are two types of buses in the network, buses with inertia and buses without inertia. Let G and L be the sets of buses with and without inertia respectively such that $|G|+|L| = |N|$. Furthermore, we use (i, j) to denote the link connecting buses i and j and assume that the graph (N, E) is directed[2] with arbitrary direction, so that if $(i, j) \in E$ then $(j, i) \notin E$. For each $j \in N$, we use $i : i \to j$ and $k : j \to k$ to denote the sets of buses that are predecessors and successors of bus j respectively. It is important to note that the form of the dynamics in (4.3)–(4.4) below is unaltered by any change in the graph ordering, and all of our results are independent of the direction. We also assume that (N, E) is connected and that:
1) Bus voltage magnitudes are $|V_j| = 1$ p.u. for all $j \in N$.
2) Lines $(i, j) \in E$ are lossless and characterized by their susceptances $B_{ij} = B_{ji} > 0$.
3) Reactive power flows do not affect bus voltage phase angles and frequencies.
These assumptions are frequently used in the literature to simplify the analysis, and usually hold at higher voltages and when the voltage within the system is tightly controlled.

The rate of change of frequency can then be represented using swing equations, while power must be conserved at each of the buses without inertia. This motivates the following system dynamics (e.g. [113])

$$\dot{\eta}_{ij} = \omega_i - \omega_j,\ (i, j) \in E, \tag{4.3a}$$

$$M_j \dot{\omega}_j = -p_j^L + p_j^M - (d_j^c + d_j^u) - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij},\ j \in G, \tag{4.3b}$$

[2] It should be noted that the power transfer between buses is bidirectional, i.e. if p_{ij} is the power transfer from i to j then the power transfer from j to i is $-p_{ij}$. The notion of a directed graph is only used here to facilitate notation so that a single variable, p_{ij} or p_{ji}, is defined for each pair of buses.

ω_j	frequency at bus j
η_{ij}	power angle difference between bus i and bus j
p_j^M	mechanical power injection at bus j
d_j^c	controllable load at bus j
d_j^u	uncontrollable frequency dependent load at bus j
p_{ij}	power transfer from bus i to bus j
B_{ij}	line susceptance between bus i and bus j
p_j^L	step change in uncontrollable demand at bus j
$x^{M,j}$	internal states of generation dynamics at bus j
$x^{c,j}$	internal states of controllable load dynamics at bus j
$x^{u,j}$	internal states of uncontrollable frequency dependent load dynamics at bus j

Table 4.1. Notation used in the system model (4.3)–(4.4). Note that variables ω_j, p_j^M, d_j^c, d_j^u, p_j^L denote deviations from corresponding nominal values. Also by internal states we refer to the states in the state space representation of the differential equations representing the dynamics (details can be found in Sections 4.3 and 4.4).

$$0 = -p_j^L - (d_j^c + d_j^u) - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij},\ j \in L, \tag{4.3c}$$

$$p_{ij} = B_{ij} \sin \eta_{ij} - p_{ij}^{nom},\ (i,j) \in E. \tag{4.3d}$$

In system (4.3) p_j^M, ω_j, d_j^c and d_j^u are time-dependent variables representing, respectively, deviations from a nominal value[3] for the following quantities: the mechanical power injection to the generator bus j, the frequency at any bus j, the controllable load and uncontrollable frequency-dependent load present at any bus j. The variables η_{ij} and p_{ij} represent, respectively, the power angle difference[4] and the deviation from a nominal value p_{ij}^{nom} for the power transmitted from bus i to bus j. The constant $M_j > 0$ denotes the inertia at bus j. The variable p_j^L denotes the deviation from a nominal value of a step change in the uncontrollable demand or generation at bus j. It should be noted that although the system frequency is the same at each bus at equilibrium, it can be different during the transient behaviour after a disturbance. This is a feature incorporated within our model.

To investigate decentralized control schemes for generation and controllable load based upon local measurements of the frequency alone, we close the loop in (4.3) by determining each of p_j^M, d_j^c, and d_j^u as outputs from independent systems of the

[3]A nominal value of a variable is defined as its value at an equilibrium of (4.3) with frequency equal to the nominal value of 50Hz (or 60Hz).

[4]The quantities η_{ij} represent the phase differences between buses i and j, given by $\theta_i - \theta_j$. The angles themselves must also satisfy $\dot{\theta}_j = \omega_j$ at all $j \in N$, however, we omit this equation in (4.3) since the power transfers p are functions only of the phase differences.

form in Section 4.3 with inputs given by the negative of the local frequency,

$$\begin{aligned} \dot{x}^{M,j} &= f^{M,j}(x^{M,j}, -\omega_j), \\ p_j^M &= g^{M,j}(x^{M,j}, -\omega_j), \end{aligned} \qquad j \in G, \tag{4.4a}$$

$$\begin{aligned} \dot{x}^{c,j} &= f^{c,j}(x^{c,j}, -\omega_j), \\ d_j^c &= g^{c,j}(x^{c,j}, -\omega_j), \end{aligned} \qquad j \in N, \tag{4.4b}$$

$$\begin{aligned} \dot{x}^{u,j} &= f^{u,j}(x^{u,j}, -\omega_j), \\ d_j^u &= g^{u,j}(x^{u,j}, -\omega_j), \end{aligned} \qquad j \in N. \tag{4.4c}$$

For convenience in the notation we collect[5] the variables in (4.4) into the vectors $x^M = [x^{M,j}]_{j\in G}$, $x^c = [x^{c,j}]_{j\in N}$, and $x^u = [x^{u,j}]_{j\in N}$. These quantities represent the internal states of the dynamical systems[6] used to update the desired outputs p_j^M, d_j^c, and d_j^u. The variables p_j^M and d_j^c are controllable, so we have freedom in our analysis to design certain properties of the dynamics in (4.4a) and (4.4b). By contrast, d_j^u represents uncontrollable load and the dynamics in (4.4c) are thus fixed. Note that the systems in (4.4) can be heterogeneous and of arbitrary dimension.

Throughout the chapter we aim to characterize broad classes of dynamics associated with generation and demand, so that *stability* and *optimality* can be guaranteed for the equilibrium points of the overall interconnected system (4.3)–(4.4).

4.4.2 Equilibrium analysis

We now define the equilibria[7] of the system (4.3)–(4.4).

Definition 4.1 *The constants* $(\eta^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*})$ *define an equilibrium of the system* (4.3)–(4.4) *if the following hold*

$$0 = \omega_i^* - \omega_j^*,\ (i,j) \in E, \tag{4.5a}$$

$$0 = -p_j^L + p_j^{M,*} - (d_j^{c,*} + d_j^{u,*}) - \sum_{k:j\to k} p_{jk}^* + \sum_{i:i\to j} p_{ij}^*,\ j \in G, \tag{4.5b}$$

[5]Each local variable (e.g. $x^{M,j}$) is a vector with multiple components.

[6]Note that since we allow general classes of dynamics for p^M and d^u, system damping can be incorporated as part of these dynamics.

[7]The interconnected system (4.3)–(4.4) could in general have multiple equilibria. It should be noted that the assumption in Section 4.3 of having a unique equilibrium point when the input is constant is a condition on the individual subsystems representing loads and generation and does not preclude their interconnection from having multiple equilibrium points.

$$0 = -p_j^L - (d_j^{c,*} + d_j^{u,*}) - \sum_{k:j\to k} p_{jk}^* + \sum_{i:i\to j} p_{ij}^*,\ j \in L, \tag{4.5c}$$

$$x^{M,j,*} = k_{x^{M,j}}(-\omega_j^*),\ j \in G, \tag{4.5d}$$

$$x^{c,j,*} = k_{x^{c,j}}(-\omega_j^*),\ x^{u,j,*} = k_{x^{u,j}}(-\omega_j^*),\ j \in N \tag{4.5e}$$

where the quantities in (4.5b) *and* (4.5c) *are given by*

$$p_{ij}^* = B_{ij} \sin \eta_{ij}^* - p_{ij}^{nom},\ (i,j) \in E, \tag{4.5f}$$

$$p_j^{M,*} = k_{p_j^M}(-\omega_j^*),\ j \in G, \tag{4.5g}$$

$$d_j^{c,*} = k_{d_j^c}(-\omega_j^*),\ d_j^{u,*} = k_{d_j^u}(-\omega_j^*),\ j \in N. \tag{4.5h}$$

We call (4.5) *the equilibrium conditions for the system* (4.3)–(4.4)*.*

Remark 4.1 *For any equilibrium with a given frequency value ω^*, the uniqueness of the output in the definition of the static input-state characteristic map in Section 4.3 means that the values of $p^{M,*}$, $d^{c,*}$, and $d^{u,*}$ are all guaranteed to be unique. By contrast, there can in general be multiple choices of η^* and p^* such that the equilibrium equations* (4.5) *remain valid, and therefore the equilibrium power transfers p_{ij}^* need not be unique*[8]*. It can be shown that they become unique under prescribed network structures, such as tree topologies.*

Remark 4.2 *From a control perspective, system* (4.3)–(4.4) *is one with input p^L and states $(\eta, \omega, x^M, x^c, x^u)$. The rest of the variables, (p, p^M, d^c, d^u), are functions of the states of the system and are used for notational convenience*[9]*.*

Throughout the remainder of the chapter we suppose that there exists some equilibrium of (4.3)–(4.4) as defined in Definition 4.1. We let $(\eta^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*})$ denote any such equilibrium and use $(p^*, p^{M,*}, d^{c,*}, d^{u,*})$ to represent the corresponding quantities defined in (4.5f)–(4.5h). We now impose a security constraint on the equilibrium power flows generated (see e.g. [114]).

[8] Note that at equilibrium the relation $0 = s^* + Qp^*$ holds where $s^* = \big[[s^{G,*}]_{j\in G}\ [s^{L,*}]_{j\in L}\big]$ is the vector of power supply variables defined in (4.6), $p^* = [p_{ij}^*]_{(i,j)\in E}$ is the vector of power transfers, and Q is a matrix with entries that take values $0, 1$ or -1 (the incidence matrix of the underlying graph). The nonunique values of p^* occur when Q has a nontrivial nullspace, which can be the case when the underlying graph has cycles. It is shown by Lemma 4.1 in Appendix A how an additional condition on η results to unique equilibrium values for any given steady state value of frequency.

[9] See also Remark 2.1.

Assumption 4.1 $|\eta^*_{ij}| < \frac{\pi}{2}$ *for all* $(i,j) \in E$.

Note that the static input-output characteristic maps $k_{p^M_j}$, $k_{d^c_j}$, and $k_{d^u_j}$ relating power generation/demand with frequency, as defined in (4.2), completely characterize the effect of the dynamics (4.4) on the behaviour of the power system (4.3) at equilibrium. In our analysis, we will consider a class of dynamics within (4.4) for which any such equilibrium point is *asymptotically stable*. Within this class, we then consider appropriate conditions on these characteristic maps such that the values of the variables defined in (4.5g)–(4.5h) are *optimal* for an appropriately constructed network optimization problem.

4.4.3 Combined passive dynamics from generation and load

In terms of the outputs in (4.4), we define the net supply variables

$$s^G_j = p^M_j - (d^c_j + d^u_j),\ j \in G, \tag{4.6a}$$

$$s^L_j = -(d^c_j + d^u_j),\ j \in L. \tag{4.6b}$$

Correspondingly, their values at equilibrium can be written as $s^{G,*}_j = p^{M,*}_j - (d^{c,*}_j + d^{u,*}_j)$ and $s^{L,*}_j = -(d^{c,*}_j + d^{u,*}_j)$.

The variables defined in (4.6) evolve according to the dynamics in (4.4). Consequently, s^G_j and s^L_j can be viewed as outputs from these combined dynamical systems with inputs $-\omega_j$.

We now introduce a notion of passivity for systems of the form (4.1), which we will use for the dynamics of the supply variables defined in (4.6) to prove our main stability results.

Definition 4.2 *The system* (4.1) *with input u, state x, and output y, is said to be locally input strictly passive about the constant input values $\bar{u}$ and the constant state values $\bar{x}$ if there exist open neighbourhoods U of $\bar{u}$ and X of $\bar{x}$ and a continuously differentiable, positive semidefinite function $V(x)$ (the storage function) such that, for all $u \in U$ and all $x \in X$, $\dot{V}(x,u) \leq (u-\bar{u})^T(y-\bar{y}) - \phi(u-\bar{u})$, where ϕ is a positive definite function and $\bar{y} = k_y(\bar{u})$. If the regions U and X are the whole of $\mathbb{R}$ and $\mathbb{R}^m$ respectively, we say that system* (4.1) *is globally input strictly passive about the equilibrium point specified.*

Remark 4.3 *The storage function can be interpreted as a form of internal energy of the system. The passivity property can easily be checked for static nonlinearities,* *and one of its important features is that for a general linear system it can be verified by means of computationally efficient methods. In particular, it follows from the KYP Lemma [49] that passivity of a linear system is equivalent to the feasibility of a linear matrix inequality (LMI), i.e. a computationally efficient convex optimization problem from which the storage function can also be constructed. Passivity can also be checked for linear systems from the positive realness of the corresponding transfer function, using the fact that positive realness is equivalent for stable systems to the frequency response function lying in the right half-plane. Various examples involving nonlinear and linear dynamics will be discussed in Section 4.6.*

We suppose that the supply dynamics (4.6) at each bus satisfy the local passivity condition in Definition 4.2. This is a decentralized condition, since it involves only the local supply dynamics at each bus.

Assumption 4.2 *Each of the systems defined in* (4.4) *with inputs* $-\omega_j$ *and outputs given by* (4.6a) *and* (4.6b) *respectively are locally input strictly passive about their equilibrium values* $-\omega_j^*$ *and* $(x^{M,j,*}, x^{c,j,*}, x^{u,j,*})$*, in the sense described in Definition 4.2.*

Remark 4.4 *It should be noted that the passivity property is assumed without specifying the precise form of the systems, which permits the inclusion of a broad class of generation and load dynamics. Also the fact that passivity is assumed only for the net supply dynamics, rather than for the generation and load dynamics individually, can permit the analysis of systems incorporating dynamics that are not individually passive.*

4.4.4 Optimal supply and load control

We aim to explore how the generated power and controllable loads may be adjusted to meet the step change p^L in frequency-independent load in a way that minimizes the total cost that comes from the extra power generated and the disutility of loads. We now introduce an optimization problem, which we call the optimal supply and load control problem (OSLC), that can be used to achieve this goal.

Suppose that costs $C_j(p_j^M)$ and $C_{dj}(d_j^c)$ are incurred for deviations p_j^M and d_j^c in generation and controllable load respectively. Furthermore, some additional cost is incurred due to any change in frequency which alters the uncontrollable frequency-dependent demand. We represent this by an integral cost[10] in terms of a function h_j which is determined by the dynamics in (4.4c) as

$$h_j(z) = k_{d_j^u}(-z) \text{ for all } z \in \mathbb{R}. \tag{4.7}$$

The total cost within OSLC then sums all the above costs, and the problem is to choose the vectors p^M, d^c, and d^u that minimize this total cost and simultaneously achieve power balance, while satisfying physical saturation constraints.

$$\begin{aligned}
&\underline{\text{OSLC}}: \\
&\min_{p^M, d^c, d^u} \sum_{j \in G} C_j(p_j^M) + \sum_{j \in N} \Big(C_{dj}(d_j^c) + \int_0^{d_j^u} h_j^{-1}(z)\, dz \Big) \\
&\text{subject to} \sum_{j \in G} p_j^M = \sum_{j \in N} (d_j^c + d_j^u + p_j^L), \\
&p_j^{M,min} \leq p_j^M \leq p_j^{M,max}, \forall j \in G, \\
&d_j^{c,min} \leq d_j^c \leq d_j^{c,max}, \forall j \in N,
\end{aligned} \tag{4.8}$$

where $p_j^{M,min}, p_j^{M,max}, d_j^{c,min}$, and $d_j^{c,max}$ are the bounds for generation and controllable demand respectively at bus j. The equality constraint in (4.8) represents conservation of power, i.e. that all the frequency-independent load is matched by the total generation plus all the frequency-dependent load contributions.

Remark 4.5 *The variables p^M and d^c within* (4.8) *represent the variables that can be directly controlled, while the variable d^u can be controlled only indirectly by effecting changes in the system frequencies. Therefore, we aim to specify properties of the control dynamics in* (4.4a)–(4.4b) *that ensure that the quantities p^M and d^c, along with the system frequencies, converge to values at which optimality in* (4.8) *can be guaranteed.*

Remark 4.6 *The two types of cost representations in* (4.8) *and the way these are connected with the system dynamics via* (4.7) *and* (4.9) *within Theorem 4.2 below, are mathematically equivalent. That is, it would also have been possible to represent the cost of uncontrollable frequency dependent loads as a function (rather than an integral), and represent* (4.7) *in terms of the derivative of this function. We use an*

[10] We use this alternative representation for the cost, in order to express the cost incurred as a function of properties of the system dynamics, as for uncontrollable loads no design of control system dynamics is feasible.

integral representation of the cost for uncontrollable demand in order to express the cost incurred as a function of properties of the system dynamics, as for uncontrollable loads no design of control system dynamics is feasible.

4.4.5 Additional conditions

To guarantee convergence and optimality, we will require additional conditions on the behaviour of the systems (4.3)–(4.4) and the structure of the optimization problem (4.8). The assumptions introduced are all of practical relevance, and we will see in Section 4.6 that the framework considered encompasses a number of important examples from the literature. Within the second condition we denote $\omega^G = [\omega_j]_{j\in G}$ and $\omega^L = [\omega_j]_{j\in L}$.

Assumption 4.3 *The storage functions in Assumption 4.2 have strict local minima at the points* $(x^{M,j,*}, x^{c,j,*}, x^{u,j,*})$ *for* $j \in G$ *and* $(x^{c,j,*}, x^{u,j,*})$ *for* $j \in L$ *respectively.*

Remark 4.7 *In practice, Assumption 4.3 is often trivially satisfied. For instance, if the vector fields in* (4.4) *are continuously differentiable, then by linearising about equilibrium, the KYP Lemma generates a storage function satisfying Assumption 4.3 whenever the linearised system is controllable and observable.*

Assumption 4.4 *There exists an open neighbourhood* T *of* $(\eta^*, \omega^{G,*}, x^{M,*}, x^{c,*}, x^{u,*})$ *such that at any time instant* t, $\omega^L(t)$ *is uniquely determined by the system states* $(\eta(t), \omega^G(t),\ x^M(t),\ x^c(t), x^u(t)) \in T$ *via a locally Lipschitz map* f^L *such that* $\omega^L = f^L(\eta, \omega^G,\ x^M, x^c, x^u)$.

Remark 4.8 *Assumption 4.4 is a technical assumption that is required in order for the system* (4.3)–(4.4) *to have a locally well-defined state space realization. This is needed in order to apply Lasalle's Theorem to analyze stability in the proof of Theorem 4.1 below.*

Remark 4.9 *Assumption 4.4 can often be verified by using the Implicit Function Theorem to generate decentralized algebraic conditions under which it is guaranteed to hold. For instance, Assumption 4.4 always holds if in* (4.4) *we have, for all* $j \in L$, $\frac{\partial g^{c,j}}{\partial \omega_j} + \frac{\partial g^{u,j}}{\partial \omega_j} \neq 0$ *at the equilibrium point. If the functions* $g^{c,j}$ *and* $g^{u,j}$ *have no explicit dependence on* ω_j, *satisfying* $\sum_i \frac{\partial g^{c,j}}{\partial x_i^{c,j}} \frac{\partial f_i^{c,j}}{\partial \omega_j} + \sum_i \frac{\partial g^{u,j}}{\partial x_i^{u,j}} \frac{\partial f_i^{u,j}}{\partial \omega_j} > 0$ *at the equilibrium point is also sufficient. These conditions are invoked in Section 4.7.*

Assumption 4.5 *The cost functions* C_j *and* C_{dj} *are continuously differentiable and strictly convex. Moreover, the first derivative of* $h_j^{-1}(z)$ *is nonnegative for all* $z \in \mathbb{R}$.

4.5 Main results

In this section we state our main results, with the proofs of Theorems 4.1–4.4 provided in Appendix A. Our first result shows that the set of equilibria of the system (4.3)–(4.4) for which the assumptions stated are satisfied is asymptotically attracting, while our second result demonstrates sufficient conditions for equilibrium points to be optimal for the OSLC problem (4.8). Based on these results, we can guarantee convergence to optimality of all solutions starting in the vicinity of an equilibrium. Finally, we show that the inclusion of controllable demand in our model reduces steady state frequency deviation, thereby aiding in frequency control.

Theorem 4.1 *Suppose that Assumptions 4.1–4.4 are all satisfied. Then there exists an open neighbourhood S of the equilibrium $(\eta^*, \omega^{G,*}, x^{M,*}, x^{c,*}, x^{u,*})$ such that whenever the initial conditions $(\eta(0), \omega^G(0), x^M(0), x^c(0), x^u(0)) \in S$, then the solutions of the system* (4.3)–(4.4) *converge to an equilibrium as defined in Definition 4.1.*

Remark 4.10 *It will be seen within the proof of Theorem 4.1 that ω, x^M, x^c, x^u converge to $\omega^*, x^{M,*}$, $x^{c,*}, x^{u,*}$ respectively. The phase differences η also converge to a fixed point, however, this can be different from η^* (see also Remark 4.1).*

Theorem 4.2 *Suppose that Assumption 4.5 is satisfied. If the control dynamics in* (4.4a) *and* (4.4b) *are chosen such that*

$$k_{p_j^M}(\omega^*) = [(C_j')^{-1}(\omega^*)]_{p_j^{M,min}}^{p_j^{M,max}} \tag{4.9a}$$

$$k_{d_j^c}(-\omega^*) = [(C_{dj}')^{-1}(\omega^*)]_{d_j^{c,min}}^{d_j^{c,max}} \tag{4.9b}$$

then the values $p^{M,*}$, $d^{c,*}$, and $d^{u,*}$ are optimal for the OSLC problem (4.8).

Theorem 4.3 *Consider equilibria of* (4.3)–(4.4) *with respect to which Assumptions 4.1–4.5 are all satisfied. If the control dynamics in* (4.4a) *and* (4.4b) *are chosen such that* (4.9) *holds, then there exists an open neighbourhood of initial conditions about any such equilibrium such that the solutions of* (4.3)–(4.4) *are guaranteed to converge to a global minimum of the OSLC problem* (4.8).

Remark 4.11 *Theorem 4.3 states that if the system* (4.3)–(4.4) *starts sufficiently close to any of its equilibria with respect to which Assumptions 4.1–4.5 are satisfied, then the system converges to an equilibrium point which is optimal for the OSLC* *problem* (4.8)*. The fact that* p^M *and* d^c *represent controllable quantities means that we are free to design the dynamics in* (4.4a) *and* (4.4b) *in order that the conditions* (4.9) *are satisfied. Thus, knowledge of the cost functions in the optimization problem we want to solve explicitly determines classes of dynamics which are guaranteed to yield convergence to optimal solutions.*

Below, we demonstrate that the inclusion of controllable loads with dynamics that satisfy our proposed passivity condition result to a drop in steady state frequency deviation and therefore aids in secondary frequency control.

Theorem 4.4 *Suppose that Assumption 4.5 is satisfied. If the control dynamics in* (4.4a) *and* (4.4b) *are chosen such that* (4.9) *holds, then the addition of controllable demand in primary control results in a drop in the steady state frequency deviation from its nominal value.*

4.6 Discussion

We now discuss various examples of generation and load dynamics that can fit within our framework.

As a first example, consider the model in [60], which investigates a linearised version of the system (4.3) coupled with the static nonlinearities $d^c_j = (C'_{dj})^{-1}(\omega_j)$ for the controllable demand, and with uncontrollable loads of the form $d^u_j = D_j \omega_j$. The damping constants D_j were assumed positive, the cost functions C_{dj} were taken to be strictly convex, and the mechanical power injection p^M was also assumed to be constant after a step change. It is easy to see that for such a system Assumptions 4.1–4.5 are all satisfied. Hence, this model can be analysed in the framework introduced above, thus implying optimality and stability of the equilibrium points.

The present framework can also include systems in which the generated powers satisfy any first-order dynamics as in [111], since such schemes are passive about their equilibria for arbitrary gains. For higher-order schemes, however, the dynamics for p^M are not necessarily passive, so some additional conditions are needed to ensure stability. A significant example of this can be seen in the second-order generation dynamics that are often considered in literature to model turbine-governor dynamics, e.g. [113, p. 382]. These can be described by

$$\begin{aligned} \dot{\alpha}_j &= -\frac{1}{\tau_{g,j}}\alpha_j + \frac{1}{\tau_{g,j}}p_j^c, \\ \dot{p}_j^M &= -\frac{1}{\tau_{b,j}}p_j^M + \frac{1}{\tau_{b,j}}\alpha_j, \end{aligned} \qquad j \in G \tag{4.10}$$

where α_j is the valve position of the turbine, the constants $\tau_{g,j}$ and $\tau_{b,j}$ represent lags in the dynamics of the governor and turbine respectively, and p_j^c is a static function of frequency, corresponding to droop control. Consider the case where generator damping and uncontrollable loads are modeled by

$$d_j^u = D_j\omega_j, \forall j \in N, \tag{4.11}$$

and there is no controllable demand. In [61], the condition

$$|p_j^c(\omega_j) - p_j^c(\omega_j^*)| \leq K_j|\omega_j - \omega_j^*|,\ j \in G \tag{4.12}$$

with $K_j < D_j$ was imposed.

As shown in Corollary 4.2 below, under (4.12), the overall system relating $-\omega_j$ with[11] $s_j^G = p_j^M - d_j^u$ becomes input strictly passive about the equilibrium point considered. This follows from a more general result which we now state describing the connection between the $\mathcal{L}_2$-gain of general generation dynamics and the passivity of the supply dynamics. The proofs of Proposition 4.2 and Corollary 4.2 can be found in Appendix A.

Proposition 4.1 *Let equation* (4.11) *hold and consider any generation dynamics from $-\omega_j$ to p_j^M of the form* (4.4a)*. Consider also the variable $p_j^D = p_j^M + g_j(\omega_j)$, where $g_j(\omega_j)$ is any function that is nondecreasing with respect to ω_j. Given any equilibrium, if the $\mathcal{L}_2$-gain from $(\omega_j - \omega_j^*)$ to $(p_j^D - p_j^{D,*})$ is strictly less than D_j, then the system with input $-\omega_j$ and output $s_j^G = p_j^M - d_j^u$ is globally input strictly passive about the equilibrium considered.*

Remark 4.12 *It should be noted that Proposition 4.1 holds for dynamical systems from ω to p^M of any order (not just second order) and the gain condition specified can*

[11] Note that this example could also include passive controllable demand $d_j^c(-\omega_j)$, since showing input strict passivity about equilibrium of the system with input $-\omega_j$ and output $s_j^G = p_j^M - d_j^u$ is sufficient to ensure also that the system with the same input and output $s_j^G = p_j^M - d_j^u - d_j^c$ is input strictly passive.

be verified for broad classes of nonlinear systems. For example, if the system from $(p_j^c - p_j^{c,*})$ *to* $(p_j^M - p_j^{M,*})$ *is linear with transfer function* $G(s)$*, with a nonlinearity at its input that satisfies* (4.12)*, then it can be shown [112] that the* $\mathcal{L}_2$*-gain condition in Proposition 4.1 is satisfied if* $\sup_\omega |G(j\omega)| \leq 1$*, by choosing* $g_j(\omega_j) = 0$*. Less conservative conditions can also be deduced by choosing a nonzero* g_j*, as it will be shown in the proof of Corollary 4.1 below.*

Remark 4.13 *It should also be noted that the passivity property in Proposition 4.1 (and also in Lemma 4.2 and Corollary 4.1 below) holds globally about the equilibrium point considered, i.e., for all values of the inputs and states of the system specified (see Definition 4.2).*

It is easy to show that for the dynamics (4.10), (4.11), (4.12), the condition in Proposition 4.1 is satisfied, and therefore the passivity property is satisfied (stated in Lemma 4.2 in Appendix A) thus recovering the stability condition in [61]. In fact, it can be shown that Proposition 4.1 allows also to relax the gain condition in (4.12) to a less conservative condition as stated in the corollary below.

Corollary 4.1 *Consider the generation dynamics in* (4.10) *and let equation* (4.11) *and*[12] (4.9a) *hold. Then, for any equilibrium where* (4.12) *holds with* $K_j < 1.53 D_j$*, the system with input* $-\omega_j$ *and output* $s_j^G = p_j^M - d_j^u$ *is globally input strictly passive about this equilibrium.*

Remark 4.14 *It should be noted that Corollary 4.1 allows to deduce asymptotic stability with a gain condition that is less restrictive than the condition* $K_j < D_j$ *in [61] and does not make use of any linearisation of the system model.*

Furthermore, our framework can allow us to deduce asymptotic stability under weaker conditions than those in (4.12), Corollary 4.1 and Proposition 4.1, when linear generation dynamics are considered. To see this, we consider a linearisation of the system (4.10) about equilibrium and let $\tilde{q}$ denote the deviation of any quantity q from its equilibrium value q^*. Expressing $\tilde{p}_j^M$ in the Laplace domain gives $\hat{\tilde{p}}_j^M = \frac{1}{(\tau_{g,j}s+1)(\tau_{b,j}s+1)}\hat{\tilde{p}}_j^c$. Therefore,

$$\hat{\tilde{s}}_j^G = \hat{\tilde{p}}_j^M - \hat{\tilde{d}}_j^u = \frac{1}{(\tau_{g,j}s+1)(\tau_{b,j}s+1)}\hat{\tilde{p}}_j^c + D_j(-\hat{\tilde{\omega}}_j)$$

[12]Note that in (4.9a) $C_j(.)$ is allowed to be any strictly convex function, which implies that $p^c(.)$ is a non decreasing function. It can be shown that Corollary 4.1 still holds if this is relaxed to the mild condition that the deviations of $p^c(-\omega)$ from the equilibrium point have the same sign as those of $-\omega$.

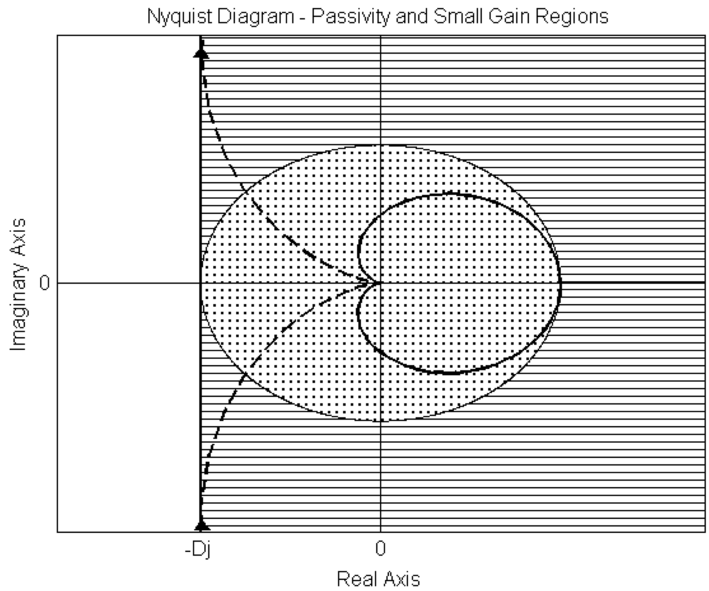

Figure 4.1. Nyquist plot for the transfer function relating $\hat{\tilde{p}}_j^M$ with $-\hat{\tilde{\omega}}_j$ when a linearisation of (4.10) about equilibrium is considered. A transfer function with a Nyquist plot within the circle satisfies the gain condition (4.12). Our approach allows the Nyquist plot to extend within the entire striped region.

$$=: H_j(s)[-\hat{\tilde{\omega}}_j],\ j \in G, \tag{4.13}$$

where $H_j(s)$ denotes the transfer function relating $-\hat{\tilde{\omega}}_j$ and $\hat{\tilde{s}}_j^G$. Since the maximum gain of the transfer function from $\tilde{p}_j^c$ to $\tilde{p}_j^M$ is 1 at $s = 0$, the condition in (4.12) constrains the Nyquist diagram of H_j to lie inside a ball with centre $(D_j, 0)$ and radius $K_j < D_j$. This is contained strictly within the right half-plane, implying the required passivity condition in Assumption 4.2. For instance, the Nyquist plot from input $-\hat{\tilde{\omega}}_j$ to output $\hat{\tilde{p}}_j^M$ can be as shown by the solid line in Fig. 4.1. However, according to our analysis any dynamics for the command signal can be permitted provided that the supply dynamics in (4.13) remain input-strictly passive. This can permit any frequency response within the striped region in Fig. 4.1, for example allowing the larger Nyquist locus shown with a dashed line. In fact, under the reasonable assumption that p_j^c has the same sign as $-\omega_j$ (i.e. negative feedback is used), it can easily be verified that the transfer function from $\tilde{p}_j^c$ to $\tilde{p}_j^M$ given by $T_j(s) = \frac{1}{(\tau_{g,j}s+1)(\tau_{b,j}s+1)}$ has a minimum real value

$$\Re\left(T_j(j\omega_j^M)\right) = \frac{-\tau_{g,j}\tau_{b,j}}{(\tau_{g,j} + \tau_{b,j})^2 + 2(\tau_{g,j} + \tau_{b,j})\sqrt{\tau_{g,j}\tau_{b,j}}} \tag{4.14}$$

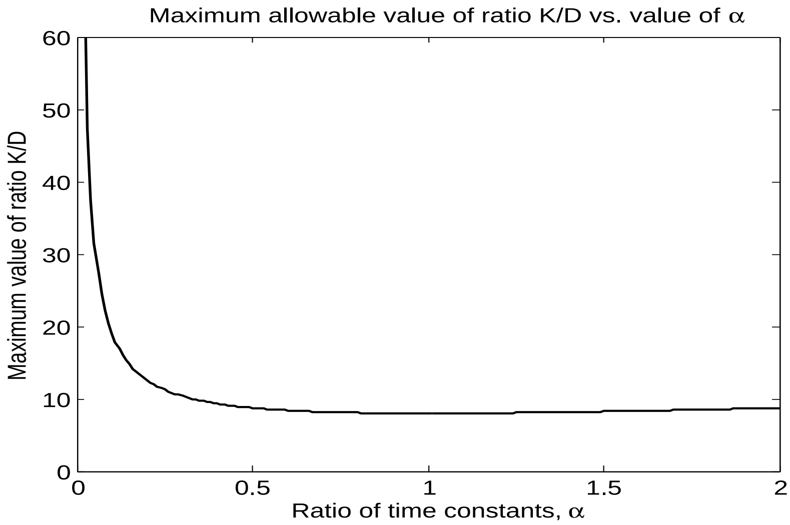

Figure 4.2. Maximum value of the ratio K_j/D_j under which the passivity property is maintained for the linearised dynamics of system (4.10)–(4.12) against the ratio of time constants $a = \tau_{b,j}/\tau_{g,j}$. The figure demonstrates that the maximum value for K_j that ensures passivity is allowed to be much higher than D_j.

at frequency $\omega_j^M = \sqrt{\frac{(\tau_{g,j}+\tau_{b,j})+\sqrt{\tau_{b,j}\tau_{g,j}}}{(\tau_{b,j}\tau_{g,j})^{3/2}}}$. Thus, the required passivity property will be maintained provided K_j multiplied by the quantity in (4.14) is strictly greater than $-D_j$. Analysis of (4.14) shows that the maximum allowable value for K_j is always at least $8D_j$ (obtained at $\frac{\tau_{b,j}}{\tau_{g,j}} = 1$) and tends to infinity as $\frac{\tau_{b,j}}{\tau_{g,j}} \to 0$ (which corresponds to a first order system) as depicted in Figure 4.2. This shows that the stability guarantees can be preserved under significantly larger gains K_j than the damping coefficients D_j. Therefore, our approach allows for a less conservative local stability condition for equilibrium points where a linearisation is feasible, while also allowing to consider a wider class of generation dynamics.

Note, however, that the use of stability conditions derived from the more conservative $\mathcal{L}_2$-gain condition in Proposition 4.1 would generally be expected to yield better robustness properties. Such trade-offs between gain and stability margin need to be taken into account in the design of control systems.

In order to to further illustrate the generality of our approach we consider below a 5th order model for the turbine/governor dynamics which is a more realistic model used by the power system toolbox [115]. This leads to the transfer function below relating the mechanical power p_j^M with the negative frequency deviation $-\omega_j$

$$G_j(s) = K_j \frac{1}{(1+sT_{s,j})} \frac{(1+sT_{3,j})}{(1+sT_{c,j})} \frac{(1+sT_{4,j})}{(1+sT_{5,j})}$$

Bus	Damping Coeff. (p.u.)	Droop Coeff. (p.u.)	Bus	Damping Coeff. (p.u.)	Droop Coeff. (p.u.)
21	34.8	43.5	50	34.44	132
22	28.6	119.4	51	21.46	76.8
23	7.34	25.8	54	51.24	180
24	34.8	135.6	55	59.78	210
25	26.4	117	56	21.94	91.8
26	3.42	165.6	57	17.71	84
27	24.3	123.4	60	21.74	91.8
36	30.3	110.1	61	8.32	44.4
42	18.86	82.2	79	48	198
47	15.17	154.8	80	23.8	144
48	15.17	154.8	82	19.6	91.8

Table 4.2. Droop and damping coefficients for generators in the NPCC Network.

where K_j and $T_{s,j}, T_{3,j}, T_{c,j}, T_{4,j}, T_{5,j}$ are the droop coefficient and time-constants respectively. Realistic values for these variables are provided by the toolbox data files for the turbine governor systems within the Northeast Power Coordinating Council (NPCC) network[13]. The passivity property required by our theory is satisfied if the Nyquist plot of $G_j(s) + D_j$, where D_j is the generator damping, is in the right half-plane. Figure 4.3 shows such plots for various buses with turbine governor systems in the NPCC network where this property is satisfied. In particular, 20 out of the 22 NPCC buses with generators with turbine governor systems satisfy the passivity property (for the other 2 buses the condition is satisfied when the damping coefficients are increased by 37% and 28% respectively). Hence the passivity property is satisfied by many existing droop control implementations and is therefore not restrictive. Note that the significance of this property is that it is a decentralized condition, and it therefore provides plug and play capabilities within the network when satisfied by all buses.

Furthermore, in order to investigate the condition in Proposition 4.1, we have included in Table 4.2 the values of the droop coefficients K_j and damping coefficients D_j of these generators. It can be seen that the droop coefficients are in most cases significantly larger than the corresponding damping coefficients. Therefore the condition $K_j < D_j$ that follows from Proposition 4.1 is not satisfied[14] and large reductions in the feedback gain are needed to enforce it.

Finally, it should be noted that our analysis could be relevant to analyse stability

[13]The data can be found in the Power System Toolbox file `datanp48` that provides parameter values for the NPCC 48 machine system.

[14]Buses 23 and 54 have two generators connected to them in the NPCC model. The table shows the parameters associated with one of the generators at each bus, but it should be noted that Proposition 4.1 is also not satisfied when the aggregate bus dynamics are taken into account.

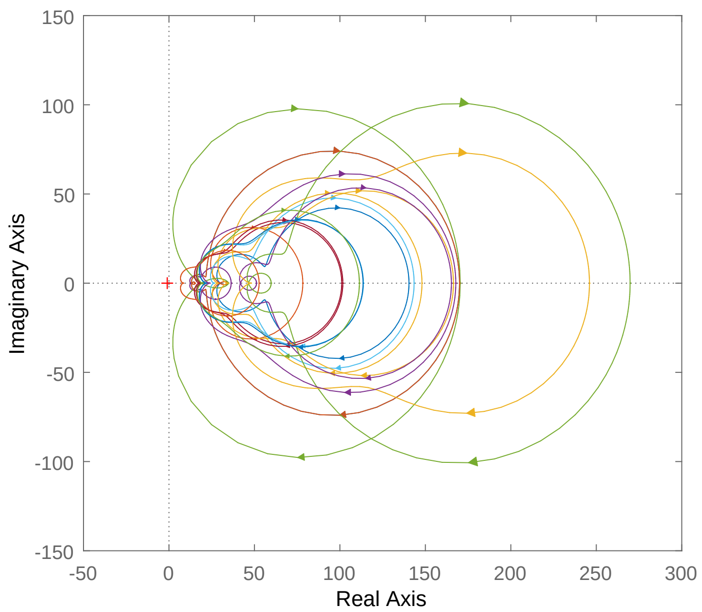

Figure 4.3. Nyquist plots of the transfer functions relating s_j with $-\omega_j$ for buses with turbine governor systems in the NPCC network where the passivity property is satisfied.

and optimality when changes in either generation or demand occur. For the case of persistent disturbances, due to e.g. renewable generation, our framework could be relevant when the timescale of those disturbances is slower than the timescale needed for the primary frequency control dynamics to reach equilibrium, i.e. typically longer than a few seconds. For faster disturbances, our analysis is also significant in the sense that lack of stability guarantees, e.g. due to insufficient damping in the system, is likely to lead to an amplification of these fluctuations within the network. Also a very conservative design (e.g. due to very small droop control gains), will lead to a system that is very slow in its response to disturbances.

4.7 Simulations on IEEE bus systems

4.7.1 Simulation on the IEEE 68-bus transmission system

In this section we illustrate our results through applications on the IEEE New York / New England 68-bus interconnection system [116], simulated using the Power System Toolbox [115]. This is more detailed and realistic than our analytical model, including line resistances, a DC12 exciter model, power system stabilizer (PSS), and

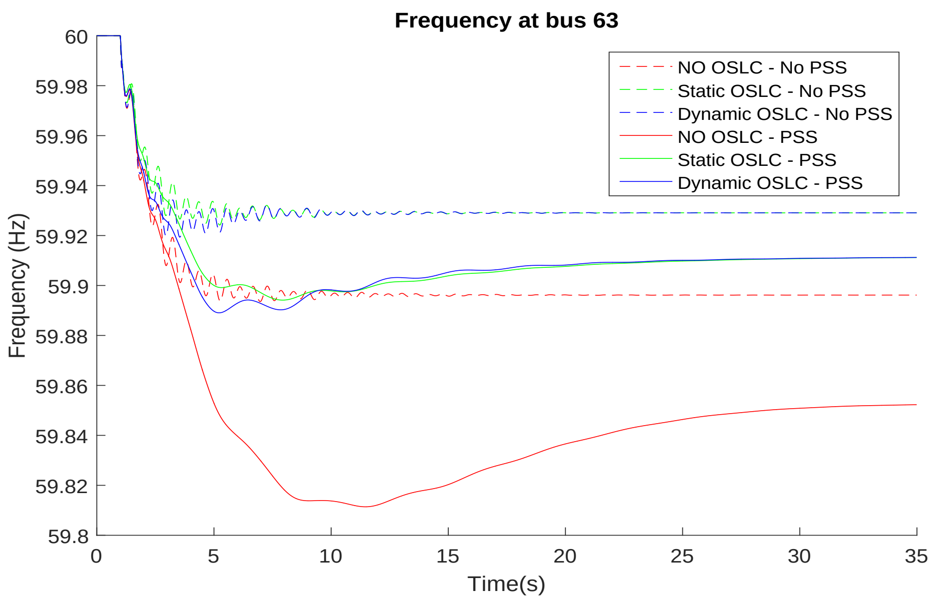

Figure 4.4. Frequency at bus 63 in six cases: (i) no OSLC, no PSS, (iii) Dynamic OSLC, no PSS, (iv) no OSLC, with PSS, (v) Static OSLC, with PSS, (vi) Dynamic OSLC, with PSS.

a subtransient reactance generator model. A similar model without PSS is used for comparison[15].

The test system contains 52 load buses serving different types of loads including constant active and reactive loads. The overall system has a total real power of 16.41GW. For our simulation, we added three loads on units 2, 9, and 17, each having a step increase of magnitude 1 p.u. (base 100MVA) at $t = 1$ second. We allow controllable demand on 34 load buses with loads controlled every 10ms. The disutility function for the aggregate load at each bus is d_j^c is $C_{dj}(d_j^c) = \frac{1}{2}\alpha_j(d_j^c)^2$. Cost coefficients α_j were selected such that the power allocated between total generation and controllable demand would be roughly equal, as suggested in [42]. The selected values were[16] $\alpha_j = 4$ for load buses 1-10 and $\alpha_j = 2$ for the rest.

Consider the static and dynamic[17] control schemes given by[18] $d_j^c = (C'_{dj})^{-1}(\omega_j)$,

[15]The details of the simulation models with or without PSS can be found in the Power System Toolbox data files data16m and data16em respectively.

[16]The values of α_j are given throughout the chapter in units consistent with the frequency measured in Hz and power in per unit. Dividing by 60 gives the value of α with the frequency also in per unit.

[17]The dynamic control scheme corresponds to cases where there is a lag in the response of the loads, or cases where low pass filtering is introduced in the control policy to avoid changes in the demand due to faster variations in grid frequency.

[18]Note that both of these are input strictly passive about the equilibria in the presence of arbitrarily small frequency damping, and both satisfy Assumption 4.4 (using respectively the conditions in Remark 4.9).

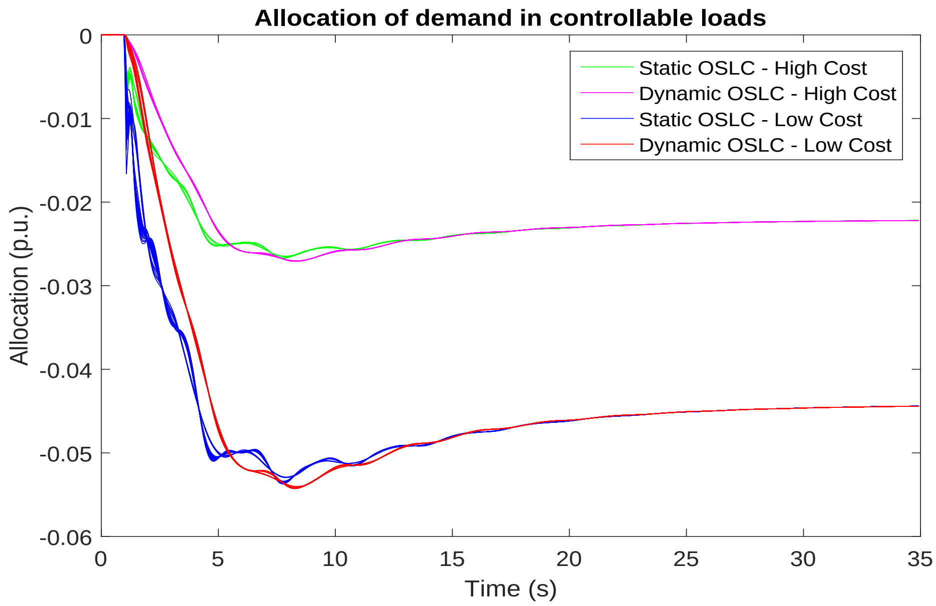

Figure 4.5. Power allocation among controllable loads with non-equal cost coefficients in two cases: (i) Static OSLC, (ii) Dynamic OSLC.

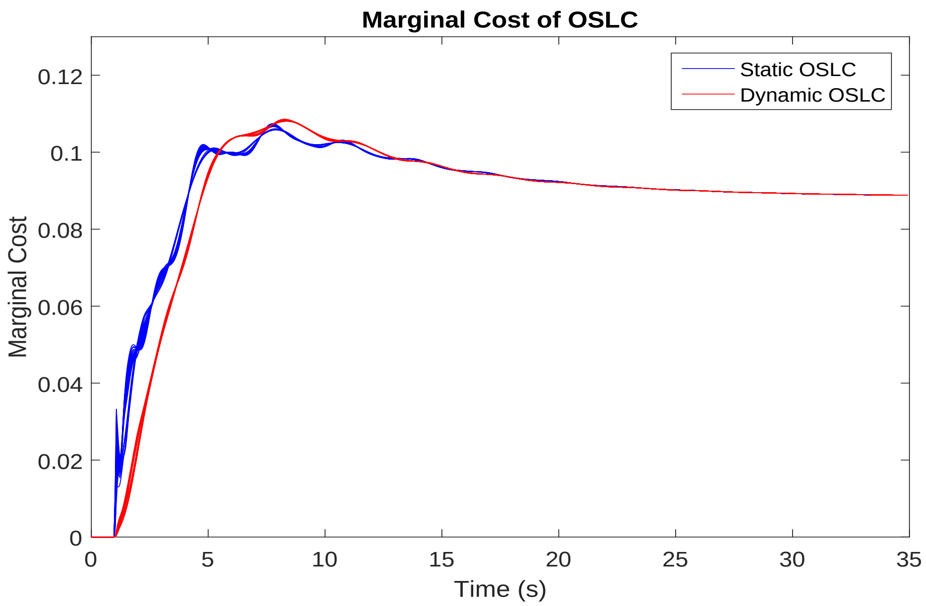

Figure 4.6. Marginal costs C'_{dj} of controllable loads with non-equal cost coefficients, in two cases: (i) Static OSLC, (ii) Dynamic OSLC.

$j \in N$, and $\dot{d}_j^c = -(C_j'(d_j^c) - \omega_j)$, $j \in N$. We refer to the resulting dynamics as Static OSLC and Dynamic OSLC respectively. We investigate the system's behaviour under the following six cases: (i) no OSLC, no PSS, (ii) Static OSLC, no PSS, (iii) Dynamic OSLC, no PSS, (iv) no OSLC, with PSS, (v) Static OSLC, with PSS, (vi)

Dynamic OSLC, with PSS. The pre-disturbance conditions of the simulations for the bus voltages, net injections and power transfers are given in Appendix B.

The frequency dynamics for bus 63 are shown in Fig. 4.4. From Fig. 4.4, we observe that whether or not PSS is used, the presence of OSLC results in a drop in steady state frequency deviations. Furthermore, we see that the overshoot is significantly less when OSLC is used. The responses for Static and Dynamic OSLC have no significant differences and converge to the same exact value at steady state. However, Dynamic OSLC appears to give a larger overshoot than Static OSLC. In all cases, the voltage deviation was less than 0.015 p.u., showing that the constant voltage assumption is reasonable. In Fig. 4.5 we also observe a higher power allocation at the load buses whose cost coefficients take the lower value $\alpha_j = 2$ than at those with $\alpha_j = 4$. This demonstrates that the power allocation among controllable loads depends upon the loads' respective cost coefficients of demand response. This behaviour could be beneficial if a prescribed allocation were desirable, as then the load dynamics could be designed such that the cost coefficients chosen yield the desired allocation. Furthermore, as shown in Fig. 4.6, the marginal costs at each controlled load converge to the same value. This illustrates the optimality in division among loads, as equality of marginal costs is the optimality condition for (4.8) when the allocations do not saturate.

To study how the amount of controllable demand affects the grid frequency, we repeated the simulation on the system with PSS and static load control schemes, varying this time the number of controllable loads. The resulting time responses are shown on Figure 4.7. From there, it can be seen that an increase in the amount of controllable loads results to a reduced frequency deviation at all times. This therefore results in nadir and steady state values of frequency that are closer to the nominal frequency.

Finally, to investigate the system's robustness, we then introduced delays to account for the time between the arrival of the frequency signal and the response of the controllable demand. The simulation was repeated with 0.1 p.u. loads and a delay of 0.05 seconds. Furthermore, all cost coefficients were set to $\alpha_j = 1$. Dynamic OSLC was seen to offer improved robustness to the time-delay relative to Static OSLC, since the first converged both with and without PSS whereas the latter became unstable in both cases. This illustrates how appropriate higher order dynamics can have improved robustness properties. The simulation results for Dynamic OSLC are depicted in Fig. 4.8. This enhanced robustness to delays can be explained with the help of Fig. 4.9. The figure shows the Nyquist plots of the transfer functions

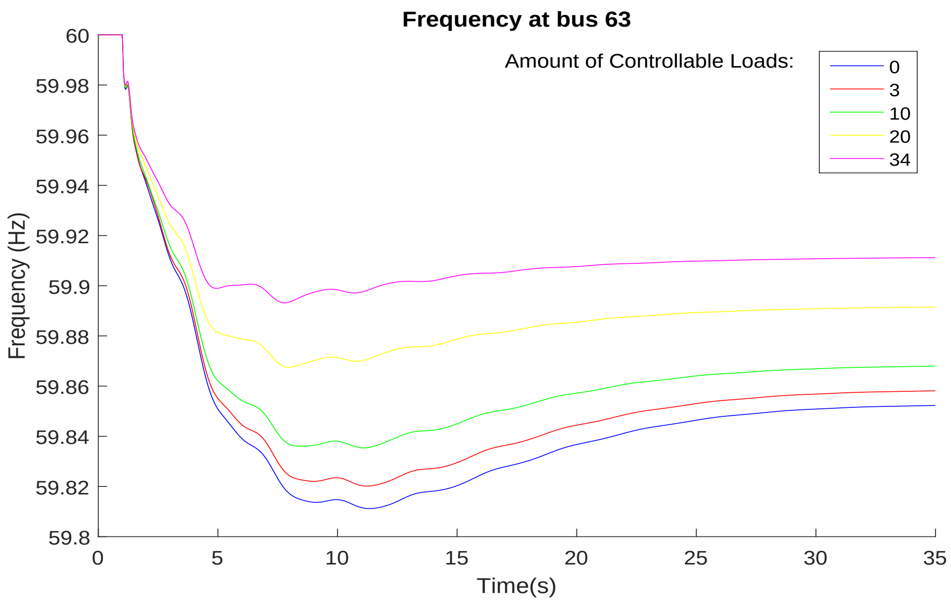

Figure 4.7. Frequency at bus 63 with an increasing number of controllable loads.

relating the increments from equilibrium of d_j^c and $-\omega_j$ for Static and Dynamics OSLC schemes, when input is delayed and both schemes multiplied by a positive gain $K > D_j$, where D_j is the damping coefficient at bus j, as in (4.11). The delayed Dynamic OSLC (dashed line) maintains the passivity property of the bus dynamics (since the Nyquist plot remains to right side of $-D_j$, as was previously discussed in section 4.6). On the other hand, the delayed Static OSLC (solid line) does not, explaining why the latter might be expected to become unstable.

4.7.2 Simulations on the IEEE 37-bus distribution system

To illustrate the validity of our stability and optimality results on a lower voltage network, we simulated an appropriately modified balanced[19] version of the IEEE 37-bus distribution system [117] using the Power System Toolbox. The test system is a feeder in California, with a 4.8kV operating voltage, fed by a big power system from one particular bus, which is modelled as an infinite bus. The system simulated is more realistic than our analytical model, and includes line resistances and reactive power as well as different types of balanced loads, as constant active and reactive power, constant impedance and constant current loads. The overall system has total real and reactive power of 2.52MW and 1.25MVAr respectively.

[19]In particular, balanced loads were used and the simulation was carried out using the Power System Toolbox that uses single phase simulations.

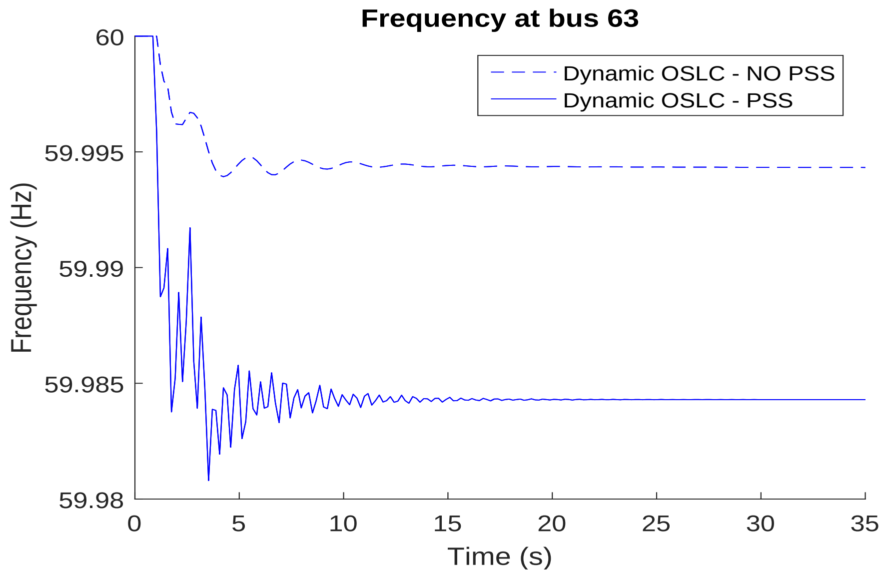

Figure 4.8. Frequency at bus 63 for Dynamic OSLC with time-delays in two cases: (i) no PSS (ii) with PSS.

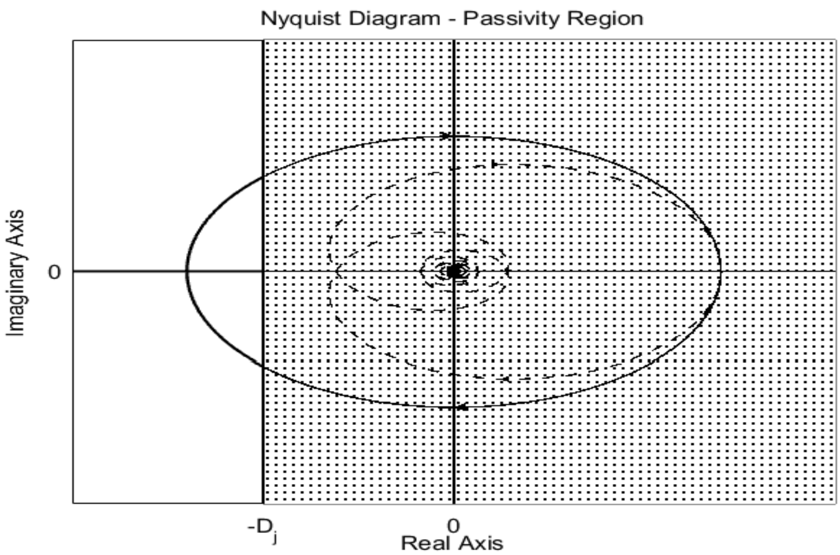

Figure 4.9. Nyquist plot of the transfer functions relating the deviations from equilibrium of d^c and $-\omega$, for Static OSLC and Dynamic OSLC linearised about equilibrium, multiplied by a constant $K > D_j$ and with an input delay also included. The figure shows that only the Dynamic delayed system (dashed line) remains on the right of $-D_j$, maintaining the passivity property of the bus dynamics when the damping coefficient is D_j as in (4.11). The Static system (solid line) extends to the left of $-D_j$, hence the passivity property is lost.

In order to examine the behaviour of controllable loads in a distribution system when a sudden change in the rest of the power grid occurs, we added a disturbance in the form of a step increase in load at the infinite bus at t=1 second, which results

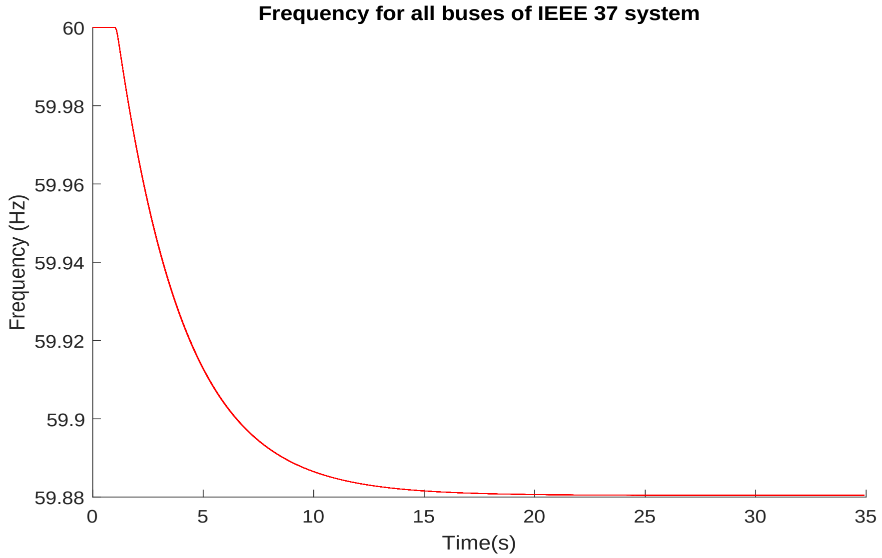

Figure 4.10. Frequency at all buses for the IEEE 37-bus distribution system.

in a change in the grid frequency. We allowed controllable loads on 24 buses with loads controlled every 10ms. Furthermore, we used the quadratic disutility function $C_{dj}(d_j^c) = \frac{1}{2}\alpha_j(d_j^c)^2$ with cost coefficient values of $a_j = 4$ for load buses 712-714 and $a_j = 2$ for the rest. Fig. 6 shows the transient behaviour when Dynamic OSLC is used for the load control schemes (the response when Static OSLC is used is very similar).

The frequency at all buses is shown in Fig. 4.10 from where the stability of the system is demonstrated. The voltage deviation is larger than that in the IEEE 68-bus network simulation taking values up to 0.022 p.u., but still relatively small. In Fig. 4.11 we observe a higher power allocation at the load buses with the lower cost coefficient $\alpha_j = 2$ than those with the higher cost coefficient $\alpha_j = 4$. From Fig. 4.12 we can also see that, as in the IEEE 68-bus simulation, the marginal costs of all controllable loads converge to the same value.

4.8 Conclusion

We have considered the problem of designing distributed generation and demand control schemes for primary frequency regulation in power networks such that asymptotic stability is guaranteed while ensuring optimality of power allocation. We have presented a network passivity framework which provides a systematic method to show stability over a broad class of generation and load dynamics. Furthermore, we have derived steady state conditions for the generation and controllable demand control schemes that ensure that the power generated/consumed is the solution to

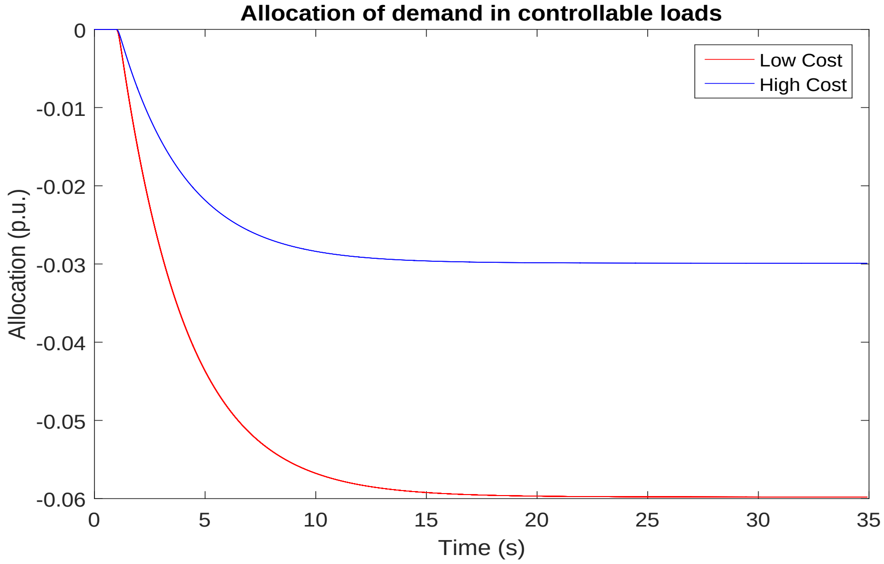

Figure 4.11. Power allocation among controllable loads with non-equal cost coefficients for Dynamic OSLC.

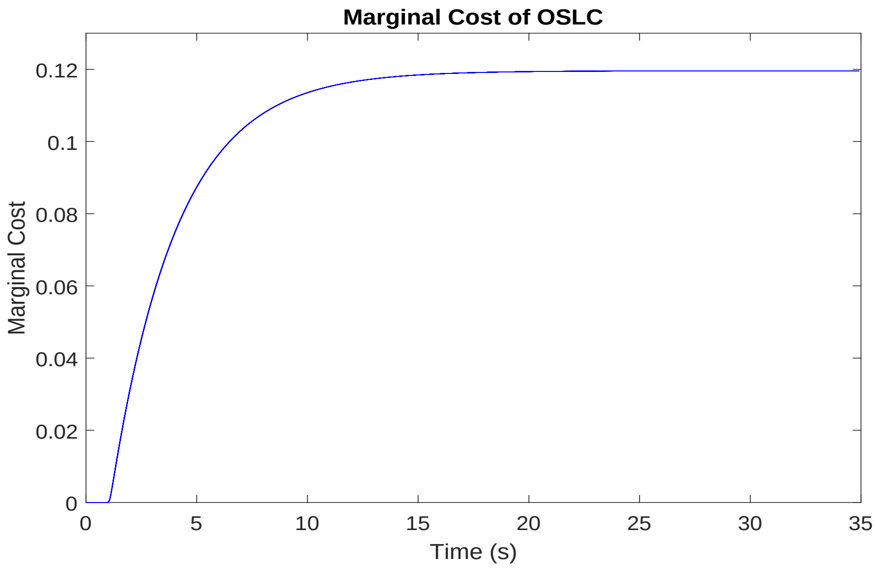

Figure 4.12. Marginal costs C'_{dj} of controllable loads with non-equal cost coefficients for Dynamic OSLC.

an appropriately constructed network optimization problem, thus allowing fairness in power allocation to be guaranteed. In addition, under some minor assumptions, we have shown that the inclusion within the model of controllable demand has a positive effect also on secondary control, decreasing the steady state deviation in frequency. Simulations on the IEEE 68-bus and IEEE 37-bus systems verify our results.

Appendix A

In this appendix we prove our main results, Theorems 4.1–4.3.

Proof of Theorem 4.1: We will use the dynamics in (4.3) together with the passivity conditions in Assumption 4.2 to define a Lyapunov function for system (4.3)–(4.4).

Firstly, we consider $V_F(\omega^G) = \frac{1}{2}\sum_{j\in G} M_j(\omega_j - \omega_j^*)^2$. The time-derivative along the trajectories of (4.3)–(4.4) is then $\dot{V}_F = \sum_{j\in G}(\omega_j - \omega_j^*)(-p_j^L + s_j^G - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij}) + \sum_{j\in L}(\omega_j - \omega_j^*)(-p_j^L + s_j^L - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij})$, by substituting (4.3b) for $\dot{\omega}_j$ for $j \in G$ and adding the final term, which is equal to zero by (4.3c). Subtracting the product of $(\omega_j - \omega_j^*)$ with each term in (4.5b) and (4.5c), we get

$$\dot{V}_F = \sum_{j\in G}(\omega_j - \omega_j^*)(s_j^G - s_j^{G,*}) + \sum_{j\in L}(\omega_j - \omega_j^*)(s_j^L - s_j^{L,*}) + \sum_{(i,j)\in E}(p_{ij} - p_{ij}^*)(\omega_j - \omega_i), \tag{4.15}$$

using in the final term the equilibrium condition (4.5a).

Additionally, consider $V_P(\eta) = \sum_{(i,j)\in E} B_{ij} \int_{\eta_{ij}^*}^{\eta_{ij}} (\sin\phi - \sin\eta_{ij}^*)\, d\phi$. Using (4.3a) and (4.3d), the time-derivative equals

$$\dot{V}_P = \sum_{(i,j)\in E} B_{ij}(\sin\eta_{ij} - \sin\eta_{ij}^*)(\omega_i - \omega_j) = \sum_{(i,j)\in E}(p_{ij} - p_{ij}^*)(\omega_i - \omega_j). \tag{4.16}$$

Now, from Assumption 4.2 and the definition of input strict passivity from Section 4.3, it follows that for each $j \in G$, there exist open neighbourhoods U_j^G of ω_j^* and X_j^G of $(x^{M,j,*}, x^{c,j,*}, x^{u,j,*})$ and a continuously differentiable, positive semidefinite function $V_j^G(x^{M,j}, x^{c,j}, x^{u,j})$ such that

$$\dot{V}_j^G \le ((-\omega_j) - (-\omega_j^*))(s_j^G - s_j^{G,*}) - \phi_j^G((-\omega_j) - (-\omega_j^*)) \tag{4.17}$$

for all $\omega_j \in U_j^G$ and all $(x^{M,j}, x^{c,j}, x^{u,j}) \in X_j^G$. Similarly, for each $j \in L$, there exist open neighbourhoods U_j^L of ω_j^* and X_j^L of $(x^{c,j,*}, x^{u,j,*})$ and a continuously differentiable, positive semidefinite function $V_j^L(x^{c,j}, x^{u,j})$ such that

$$\dot{V}_j^L \le ((-\omega_j) - (-\omega_j^*))(s_j^L - s_j^{L,*}) - \phi_j^L((-\omega_j) - (-\omega_j^*)) \tag{4.18}$$

for all $\omega_j \in U_j^L$ and all $(x^{c,j}, x^{u,j}) \in X_j^L$. In (4.17) and (4.18), ϕ_j^G and ϕ_j^L are positive definite functions.

Based on the above, we define the function

$$V(\eta, \omega^G, x^M, x^c, x^u) = V_F(\omega^G) + V_P(\eta) + \sum_{j\in G} V_j^G(x^{M,j}, x^{c,j}, x^{u,j}) + \sum_{j\in L} V_j^L(x^{c,j}, x^{u,j}).$$

By (4.15) and (4.16), $\dot{V} = \sum_{j\in G}[(\omega_j - \omega_j^*)(s_j^G - s_j^{G,*}) + \dot{V}_j^G] + \sum_{j\in L}[(\omega_j - \omega_j^*)(s_j^L - s_j^{L,*}) + \dot{V}_j^L]$. Using (4.17) and (4.18), it therefore holds that, whenever $\omega_j \in U_j^G$ and $(x^{M,j}, x^{c,j}, x^{u,j}) \in X_j^G$ for all $j \in G$ and $\omega_j \in U_j^L$ and $(x^{c,j}, x^{u,j}) \in X_j^L$ for all $j \in L$,

$$\dot{V} \leq -\sum_{j\in G} \phi_j^G((-\omega_j) - (-\omega_j^*)) - \sum_{j\in L} \phi_j^L((-\omega_j) - (-\omega_j^*)) \leq 0. \tag{4.19}$$

Clearly V_F has a strict global minimum at $\omega^{G,*}$, and V_j^G, V_j^L have strict local minima at $(x^{M,j,*}, x^{c,j,*}, x^{u,j,*})$, $(x^{c,j,*}, x^{u,j,*})$ by Assumption 4.3. Additionally, Assumption 4.1 guarantees the existence of some neighbourhood of each η_{ij}^* on which the respective integrand in the definition of V_P is increasing. Since the integrand is zero at the lower limit, η_{ij}^*, this immediately implies that V_P has a strict local minimum at η^*. Thus, V has a strict local minimum at the point $Q^* := (\eta^*, \omega^{G,*}, x^{M,*}, x^{c,*}, x^{u,*})$. We now recall Assumption 4.4. This tells us that, provided $(\eta, \omega^G, x^M, x^c, x^u) \in T$, ω^L can be uniquely determined from these quantities. Therefore, the states of the differential equation system (4.3)–(4.4) within the region T can be expressed as $(\eta, \omega^G, x^M, x^c, x^u)$. We can thus choose a neighbourhood in the coordinates $(\eta, \omega^G, x^M, x^c, x^u)$ about Q^* on which the following all hold:

1. Q^* is a strict minimum of V,
2. $(\eta, \omega^G, x^M, x^c, x^u) \in T$,
3. $\omega_j \in U_j^G$ and $(x^{M,j}, x^{c,j}, x^{u,j}) \in X_j^G$ for all $j \in G$ and $\omega_j \in U_j^L$ and $(x^{c,j}, x^{u,j}) \in X_j^L$ for all $j \in L$[20],
4. $x^{M,j}$, $x^{c,j}$, and $x^{u,j}$ all lie within their respective neighbourhoods X_0 as defined in Section 4.3.

Recalling now (4.19), within this neighbourhood V is thus a nonincreasing function of all the system states and has a strict local minimum at Q^*. Consequently, the

[20] This is possible because the requirement $\omega_j \in U_j^L$ for all $j \in L$ corresponds, by Assumption 4.4 and the continuity of (4.3)–(4.4), to requiring the states $(\eta, \omega^G, x^M, x^c, x^u)$ to lie in some open neighbourhood about Q^*.

connected component of the level set $\{(\eta, \omega^G, x^M, x^c, x^u) \colon V \leq \epsilon\}$ containing Q^* is both compact and positively invariant with respect to (4.3)–(4.4) for all sufficiently small $\epsilon > 0$. Therefore, there exists a compact positively invariant set Ξ for (4.3)–(4.4) containing Q^*.

Lasalle's Invariance Principle can now be applied with the function V on the compact positively invariant set Ξ. This guarantees that all solutions of (4.3)–(4.4) with initial conditions $(\eta(0), \omega^G(0), x^M(0), x^c(0), x^u(0)) \in \Xi$ converge to the largest invariant set within $\Xi \cap \{(\eta, \omega^G,\ x^M, x^c, x^u) \colon \dot{V} = 0\}$. We now consider this invariant set. If $\dot{V} = 0$ holds at a point within Ξ, then (4.19) holds with equality, hence by Assumption 4.2 we must have $\omega = \omega^*$. Moreover, on any invariant set on which $\omega = \omega^*$, the system equations (4.3) apply and give precisely the equilibrium conditions (4.5a), (4.5b), (4.5c), and (4.5f). Furthermore, if $\dot{V} = 0$, it follows from (4.15), (4.16), (4.17), and (4.18) that all $\dot{V}_j^G = 0$ and $\dot{V}_j^L = 0$. But $\omega = \omega^*$ implies by the definitions in Section 4.3 the convergence of (x^M, x^c, x^u) to $(x^{M,*}, x^{c,*}, x^{u,*})$, where V_j^G and V_j^L take strict local minima by Assumption 4.3. Hence, V_j^G and V_j^L must decrease along any nontrivial trajectory contradicting $\dot{V}_j^G = 0$ and $\dot{V}_j^L = 0$. Therefore, within the invariant set it holds that $(x^M, x^c, x^u) = (x^{M,*}, x^{c,*}, x^{u,*})$. Consequently, at all points of any invariant set within $\Xi \cap \{(\eta, \omega^G, x^M, x^c, x^u) \colon \dot{V} = 0\}$, we must also have $(x^M, x^c, x^u) = (x^{M,*}, x^{c,*}, x^{u,*})$. Thus, the remaining equilibrium conditions (4.5d), (4.5e), (4.5g) and (4.5h) are also satisfied. Therefore, we conclude by Lasalle's Invariance Principle that all solutions of (4.3)–(4.4) with initial conditions $(\eta(0), \omega^G(0), x^M(0), x^c(0), x^u(0)) \in \Xi$ converge to the set of equilibria defined in Definition 4.1. Finally, choosing for S any open neighbourhood of Q^* within Ξ completes the proof. ■

It should be noted that power angle differences can also be described by

$$\eta_{ij} = \theta_i - \theta_j, (i, j) \in E, \tag{4.20}$$

where θ_i denotes the power angle at bus i. When (4.20) is taken into account, it can be shown that for a given equilibrium frequency ω^* there exists a unique vector of power angle differences η^*. This follows since vector η needs to lie in a subspace of $\mathbb{R}^{|E|}$ such that there exists some θ that satisfies (4.20). This is formally stated in Lemma 4.1 below.

Lemma 4.1 *Consider the system* (4.3),(4.4),(4.20) *and equilibria that satisfy Definition 4.1 and Assumption 4.1. Then, given equilibrium frequency ω^*, the power angle differences η^* are unique.*

Proof of Lemma 4.1: Consider some equilibrium ω^* and the corresponding values for $(p^{M,*}, d^{c,*}, d^{u,*})$ that follow from (4.5g)–(4.5h). Also, let equilibrium power transfer vectors $p_1^*, p_2^* \in \mathbb{R}^{|E|}$ be such that (4.5b)–(4.5c) are satisfied. It then holds that

$$-p^L + p^{M,*} - d^{c,*} - d^{u,*} + \Gamma p_1^* = 0, \tag{4.21a}$$

$$-p^L + p^{M,*} - d^{c,*} - d^{u,*} + \Gamma p_2^* = 0, \tag{4.21b}$$

where Γ is an $|N| \times |E|$ matrix, called the incidence matrix, defined as

$$\Gamma_{ij} = \begin{cases} +1 \text{ if edge j enters bus i,} \\ -1 \text{ if edge j leaves bus i,} \\ \;\,0 \quad \text{otherwise.} \end{cases} \tag{4.22}$$

Furthermore, consider power angle difference vectors η_1^* and η_2^* that correspond to p_1^* and p_2^* that satisfy

$$p_j^* = \Phi \sin \eta_j^* - p^{nom}, j = \{1, 2\}, \tag{4.23}$$

where Φ is an $|E| \times |E|$ diagonal matrix with elements $\Phi_{ii} = B_{jk}$, where $(j,k) \in E$ is associated with edge i, and $\sin \eta^*$ is an $|E| \times 1$ vector where element j is the sinusoid of element j in η^*. It then follows, by subtracting (4.21b) from (4.21a) and substituting for (4.23) that

$$\Gamma\Phi(\sin \eta_1^* - \sin \eta_2^*) = 0 \tag{4.24}$$

and hence that

$$(\theta_1 - \theta_2)^T \Gamma\Phi(\sin \eta_1^* - \sin \eta_2^*) = 0 \tag{4.25}$$

where θ_1 and θ_2 are the angle vectors that correspond to η_1^* and η_2^* respectively. Using that $\eta = \Gamma^T \theta$, it then follows that

$$(\eta_1^* - \eta_2^*)^T \Phi(\sin \eta_1^* - \sin \eta_2^*) = 0 \tag{4.26}$$

which has only one solution at $\eta_1^* = \eta_2^*$, as follows form the positive definiteness of Φ and Assumption 4.1. ■

Remark 4.15 *Lemma 4.1 shows that when the power angle difference equation*

(4.20) *is taken into account, that for a given steady state frequency the equilibrium power transfers are unique. Hence, when* (4.20) *is taken into account, Theorem 4.1 deduces convergence to a point, rather that to a set, as follows from Lemma 4.1. Equation* (4.20) *was omitted from the initial system description since it would significantly and unnecessary complicate the presentation of the results.*

Proof of Theorem 4.2: Due to Assumption 4.5, C_j' and C_{dj}' are strictly increasing and hence invertible. Therefore all variables in (4.9) with $\bar{u} = -\omega_j^*$ are well-defined. Furthermore, Assumption 4.5 also ensures that the OSLC problem (4.8) is a convex optimization problem with a continuously differentiable cost function. Thus, a point $(\bar{p}^M, \bar{d}^c, \bar{d}^u)$ is a global minimum for (4.8) if and only if it satisfies the KKT conditions [104]

$$C_j'(\bar{p}_j^M) = -\nu - \lambda_j^+ + \lambda_j^-,\ j \in G, \tag{4.27a}$$

$$C_{dj}'(\bar{d}_j^c) = \nu - \mu_j^+ + \mu_j^-,\ j \in N, \tag{4.27b}$$

$$h_j^{-1}(\bar{d}_j^u) = \nu,\ j \in N, \tag{4.27c}$$

$$\sum_{j\in G} \bar{p}_j^M = \sum_{j \in N} (\bar{d}_j^c + \bar{d}_j^u + p_j^L), \tag{4.27d}$$

$$p_j^{M,min} \leq \bar{p}_j^M \leq p_j^{M,max},\ j \in G, \tag{4.27e}$$

$$d_j^{c,min} \leq \bar{d}_j^c \leq d_j^{c,max},\ j \in N, \tag{4.27f}$$

$$\lambda_j^+(\bar{p}_j^M - p_j^{M,max}) = 0,\ \lambda_j^-(\bar{p}_j^M - p_j^{M,min}) = 0,\ j \in G, \tag{4.27g}$$

$$\mu^+(\bar{d}_j^c - d_j^{c,max}) = 0,\ \mu^-(\bar{d}_j^c - d_j^{c,min}) = 0,\ j \in N, \tag{4.27h}$$

for some constants $\nu \in \mathbb{R}$ and $\lambda_j^+, \lambda_j^-, \mu_j^+, \mu_j^- \geq 0$. We will now show that these conditions are satisfied by the equilibrium values $(\bar{p}^M, \bar{d}^c, \bar{d}^u) = (p^{M,*}, d^{c,*}, d^{u,*})$ defined by equations (4.5g) and (4.5h).

Since C_j' and C_{dj}' are strictly increasing, we can uniquely define $\omega_j^{M,max} := -C_j'(p_j^{M,max})$, $\omega_j^{M,min} := -C_j'(p_j^{M,min})$, $\omega_j^{c,max} := C_{dj}'(d_j^{c,max})$, and $\omega_j^{c,min} := C_{dj}'(d_j^{c,min})$. Letting ω_0^* denote the common value of all ω_j^* due to (4.5a), we can, in terms of these quantities, define the nonnegative constants

$$\lambda_j^+ := (\omega_j^{M,max} - \omega_0^*)\, \mathbb{1}_{\{q:\ q \leq \omega_j^{M,max}\}}(\omega_0^*),$$

$$\lambda_j^- := (\omega_0^* - \omega_j^{M,min})\, \mathbb{1}_{\{q:\ q \geq \omega_j^{M,min}\}}(\omega_0^*),$$

$$\mu_j^+ := (\omega_0^* - \omega_j^{c,max})\,\mathbb{1}_{\{q:\, q \geq \omega_j^{c,max}\}}(\omega_0^*),$$
$$\mu_j^- := (\omega_j^{c,min} - \omega_0^*)\,\mathbb{1}_{\{q:\, q \leq \omega_j^{c,min}\}}(\omega_0^*).$$

Then, since $(C_j')^{-1}(-\omega_0^*) \geq p_j^{M,max} \Leftrightarrow \omega_0^* \leq \omega_j^{M,max}$, $(C_j')^{-1}(-\omega_0^*) \leq p_j^{M,min} \Leftrightarrow \omega_0^* \geq \omega_j^{M,min}$, $(C_{dj}')^{-1}(\omega_0^*) \geq d_j^{c,max} \Leftrightarrow \omega_0^* \geq \omega_j^{c,max}$, and $(C_{dj}')^{-1}(\omega_0^*) \leq d_j^{c,min} \Leftrightarrow \omega_0^* \leq \omega_j^{c,min}$, it follows by (4.5g), (4.5h) and (4.9) that the complementary slackness conditions (4.27g) and (4.27h) are satisfied.

Now define $\nu = \omega_0^*$. Then $(C_j')^{-1}(-\nu - \lambda_j^+ + \lambda_j^-) = (C_j')^{-1}\Big([-\omega_0^*]_{-\omega_j^{M,min}}^{-\omega_j^{M,max}}\Big) = [(C_j')^{-1}(-\omega_0^*)]_{p_j^{M,min}}^{p_j^{M,max}} = p_j^{M,*}$, by the above definitions and equations (4.5g) and (4.9). Thus, the optimality condition (4.27a) holds. Analogously, $(C_{dj}')^{-1}(\nu - \mu^+ + \mu^-) = (C_{dj}')^{-1}\Big([\omega_0^*]_{\omega_j^{c,min}}^{\omega_j^{c,max}}\Big) = [(C_{dj}')^{-1}(\omega_0^*)]_{d_j^{c,min}}^{d_j^{c,max}} = d_j^{c,*}$, by (4.5h) and (4.9), satisfying (4.27b). Additionally, (4.27c) holds as $h_j(\nu) = d_j^u$ follows from (4.5h) and (4.7).

Furthermore, summing the equilibrium conditions (4.5b) over all $j \in G$ and (4.5c) over all $j \in L$ shows that (4.27d) holds. Finally, the saturation constraints in (4.9) verify (4.27e) and (4.27f).

Thus, the values $(\bar{p}^M, \bar{d}^c, \bar{d}^u) = (p^{M,*}, d^{c,*}, d^{u,*})$ satisfy the KKT conditions (4.27). Therefore, the equilibrium values $p^{M,*}$, $d^{c,*}$, and $d^{u,*}$ define a global minimum for (4.8). ■

Proof of Theorem 4.3: If Assumptions 4.1–4.5 all hold and (4.9) is true, then all of the assumptions in both Theorems 4.1 and 4.2 are satisfied, and thus the result follows. ■

Proof of Theorem 4.4: Recalling the proof of Theorem 4.2, we know from (4.27d) and the equalities (4.9) that at any equilibrium of (4.3)–(4.4) the power balance equation

$$\sum_{j\in G}(C_j')^{-1}(\omega_0^*) + \sum_{j\in N}((C_{dj}')^{-1}(\omega_0^*) + h_j(\omega_0^*)) = -\sum_{j\in N} p_j^L \tag{4.28}$$

is satisfied, where ω_0^* denotes the common steady state value of frequency due to (4.5a). Now note that, because the nominal frequency defines an equilibrium frequency prior to the step change in load and all quantities in (4.3) denote deviations from their respective values at this nominal equilibrium, the equalities (4.7) and (4.9) imply that each term on the left-hand side in (4.28) must take the value zero at $\omega_0^* = 0$. Furthermore, Assumption 4.5 implies that the terms $(C_j')^{-1}(\omega_0^*)$ and $(C_{dj}')^{-1}(\omega_0^*)$ are all strictly increasing in ω_0^*, while each term $h_j(\omega_0^*)$ is nondecreasing in ω_0^*. Thus both the added term due to load control and the entire left-hand side in (4.28) have the same sign as ω_0^* and are strictly increasing in ω_0^*. It follows that the presence of this load control term results in a decrease in the value of ω_0^*, the steady state frequency deviation from its nominal value. ■

Proof of Proposition 4.1: The $\mathcal{L}_2$-gain condition implies[21]

$$\sqrt{\int_0^{t_1} (\tilde{p}_j^D)^2 \, dt} \leq K_j \sqrt{\int_0^{t_1} \tilde{\omega}_j^2(t) \, dt}. \tag{4.29}$$

where $K_j < D_j$ and t_1 is any positive constant. Then, input strict passivity can be shown as follows.

$$\begin{aligned}
\int_0^{t_1} \tilde{p}_j^M(t)\tilde{\omega}_j(t) \, dt &= \int_0^{t_1} \big[(\tilde{p}_j^D(t) - \tilde{g}_j(\omega_j(t))\big]\tilde{\omega}_j(t) \, dt \\
&\leq \int_0^{t_1} \tilde{p}_j^D(t)\tilde{\omega}_j(t) \, dt && (4.30a) \\
&\leq \sqrt{\int_0^{t_1} |\tilde{p}_j^D(t)|^2 \, dt \int_0^{t_1} |\tilde{\omega}_j(t)|^2 \, dt} && (4.30b) \\
&\leq \int_0^{t_1} K_j \tilde{\omega}_j(t)^2 \, dt < \int_0^{t_1} D_j \tilde{\omega}_j(t)^2 \, dt, && (4.30c)
\end{aligned}$$

where inequality (4.30a) follows from the fact that g_j is a nondecreasing[22] function of ω_j, (4.30b) from the Cauchy-Schwarz inequality, and (4.30c) from inequality (4.29) and $K_j < D_j$.

Using (4.30), it is straightforward to show that

$$\begin{aligned}
&\int_0^{t_1} D_j \tilde{\omega}_j(t)^2 dt - \int_0^{t_1} \tilde{p}_j^M(t)\tilde{\omega}_j(t) dt \\
&\qquad = \int_0^{t_1} \tilde{s}_j^G(t)(-\tilde{\omega}_j(t)) dt \geq (D_j - K_j) \int_0^{t_1} \tilde{\omega}_j(t)^2 dt \geq 0 \qquad (4.31)
\end{aligned}$$

holds for all $j \in G$. Inequality (4.31) implies input strict passivity of the system with output $\tilde{s}_j^G = \tilde{p}_j^M - \tilde{d}_j^u$ and input $-\tilde{\omega}_j$, about the equilibrium point considered, since (4.31) implies from [118, Lemma 1] the existence of a positive definite storage function V satisfying the local input strict passivity condition in Definition 4.2. ■

Before proving Corollary 4.1 we prove first a simpler result stated as Lemma 4.2 below.

[21] As in the main text, for a variable x that depends on time we use the notation $\tilde{x}$ to denote deviations from equilibrium, i.e. $\tilde{x} := x - x^*$, where x^* is the value of x at the equilibrium point mentioned in the proposition.

[22] Note that $\tilde{g}_j(\omega_j)\tilde{\omega}_j = [g_j(\omega_j) - g_j(\omega_j^*)][\omega_j - \omega_j^*] \geq 0$ since function $g_j(\omega_j)$ is nondecreasing with respect to ω_j.

Lemma 4.2 *Consider the generation dynamics in* (4.10) *and let equation* (4.11) *hold. Then, for any equilibrium where* (4.12) *holds, the system with input* $-\omega_j$ *and output* $s_j^G = p_j^M - d_j^u$ *is globally input strictly passive about this equilibrium.*

Proof of Lemma 4.2: The Lemma follows from Proposition 4.1 by showing that the $\mathcal{L}_2$-gain condition in the Proposition is satisfied with $g_j = 0$. In particular, let $T_j(s)$ be the transfer function relating $\hat{\tilde{p}}_j^c(s)$ and $\hat{\tilde{p}}_j^M(s)$ in (4.10), given by

$$T_j(s) = \frac{1}{(\tau_{g,j}s + 1)(\tau_{b,j}s + 1)}, \ j \in G.$$

It is easy to show that $\sup_\phi |T(j\phi)| = 1$, hence the system from $\tilde{p}_j^c$ to $\tilde{p}_j^M$ has $\mathcal{L}_2$-gain less than or equal to 1 (e.g. [112, p.18]). Using also equation (4.12) we thus have

$$\|\tilde{p}_j^M\|_2 \leq \|\tilde{p}_j^c\|_2 < D_j\|\tilde{\omega}_j\|_2. \tag{4.32}$$

With the choice $g_j = 0$ we have $p_j^M = p_j^D$. It hence follows from (4.32) that the $\mathcal{L}_2$-gain condition in Proposition 4.1 holds, therefore Proposition 4.1 can be used to deduce input strict passivity of the system. ■

Proof of Corollary 4.1: The proof is analogous to that of Lemma 4.1, but we additionally show that by optimizing over a class of nonzero functions g_j in Proposition 4.1, a less restrictive gain condition can be obtained.

In particular, we consider a variable p_j^D of the form

$$p_j^D = p_j^M - \hat{D}_j p_j^c$$

where $\hat{D}_j \geq 0$ is a constant that will be appropriately chosen. Note that $p_j^c(\omega_j)$ is a nonlinear function of frequency that satisfies the condition on $g_j(.)$ in Proposition 4.1, since it is equal to $k_{p_i^M}$ in (4.9a) and $(C_j')^{-1}$ is non decreasing due to the convexity of function C_j.

We now consider the system from $\tilde{p}_j^c$ to $\tilde{p}_j^D$, which has transfer function[23]

$$T_j(s) = \frac{1}{(\tau_{g,j}s + 1)(\tau_{b,j}s + 1)} - \hat{D}_j$$

Also let $\rho_j = \tau_{g,j}/\tau_{b,j}$ and note that $\|T_j\|_\infty := \sup_\phi |T_j(j\phi)|$ depends only on ρ_j and

[23]Note that $p^c(\omega)$ is still a nonlinear function of frequency and no linearization is carried out in the proof.

$\hat{D}_j$. We consider now the optimization problem

$$\sup_{\rho\geq 0}\inf_{\hat{D}_j\geq 0}\sup_{\phi\in\mathbb{R}}|T_j(j\phi)| \tag{4.33}$$

The solution to this problem is 1/1.5396 with the optimizing variables given by

$$\phi = \sqrt{\frac{\sqrt{\rho_j^3(2\hat{D}_j(\rho_j+1)-1)(2\hat{D}_j(\rho_j+1)-\rho_j)}+\rho_j^2(2\hat{D}_j-1)}{2\hat{D}_j\rho_j^3\tau_{b,j}^2}}, \tag{4.34a}$$

$$\hat{D}_j = \frac{\sqrt{\rho_j^4+14\rho_j^2+1}+\rho_j^2-6\rho_j+1}{4(\rho_j-1)^2}, \tag{4.34b}$$

$$\rho_j \to 1 \tag{4.34c}$$

where (4.34a) and (4.34b) give the optimal values of ϕ and $\hat{D}_j$ respectively for a given ρ_j, and (4.34c) gives the optimal value of ρ_j.

It hence follows from (4.33), that with $\hat{D}_j$ chosen as its optimal value in (4.34b), (4.34c), the $\mathcal{L}_2$-gain of the system from from $\tilde{p}_j^c$ to $\tilde{p}_j^D$ is less than 1/1.5396, for all values of ρ_j. Therefore if we choose the gain K in (4.12) such that $K < 1.5396D$ we have that

$$\|\tilde{p}_j^D\|_2 < \|\tilde{p}_j^c\|_2/1.5396 < D_j\|\tilde{\omega}_j\|_2. \tag{4.35}$$

Hence the $\mathcal{L}_2$-gain condition in Proposition 4.1 holds, and Proposition 4.1 can be used to deduce input strict passivity of the system. ■

Remark 4.16 *It should be noted that the bound $K < 1.5396D$ guarantees stability for any values of $\rho_j = \tau_{g,j}/\tau_{b,j}$. For specific values of ρ_j this condition can be further relaxed by considering the solution to the optimization problem* (4.33) *for the corresponding value of ρ_j considered.*

Appendix B

In this appendix we provide the pre-disturbance conditions of the simulations in Section 4.7 for the bus voltages, net injections and power transfers. It should be noted that due to losses, the net injections from Table 4.3 are different than the sum of the power flows from Table 4.4. The pre-disturbance conditions are provided by the Power System Toolbox and are calculated by solving a corresponding load flow problem.

Bus Number	Voltage (p.u.)	Phase (degrees)	Injection (p.u.)	Bus Number	Voltage (p.u.)	Phase (degrees)	Injection (p.u.)
1	1.059	6.615	2.527	35	1.014	2.533	0.000
2	1.052	8.434	0.000	36	1.042	-0.847	1.020
3	1.033	5.432	3.220	37	1.029	-6.805	60.000
4	1.006	4.314	5.000	38	1.056	8.677	0.000
5	1.007	5.254	0.000	39	1.006	-8.442	2.670
6	1.009	5.935	0.000	40	1.068	15.216	0.656
7	1.000	3.664	2.340	41	0.999	44.489	10.000
8	0.999	3.122	5.220	42	0.999	38.925	11.500
9	1.039	2.579	1.040	43	1.015	-7.606	0.000
10	1.018	8.453	0.000	44	1.014	-7.637	2.675
11	1.014	7.596	0.000	45	1.018	2.525	2.080
12	1.055	7.620	0.090	46	1.032	9.646	1.507
13	1.016	7.785	0.000	47	1.074	7.363	2.031
14	1.013	6.238	0.000	48	1.076	9.279	2.412
15	1.017	6.142	3.200	49	1.012	12.881	1.640
16	1.033	7.679	3.290	50	1.012	19.331	1.000
17	1.036	6.585	0.000	51	1.022	6.523	3.370
18	1.034	5.720	1.580	52	0.993	38.592	24.700
19	1.050	12.274	0.000	53	1.045	10.853	-2.500
20	0.990	10.841	6.800	54	0.980	14.411	-5.450
21	1.033	10.314	2.740	55	0.983	16.440	-6.500
22	1.050	14.997	0.000	56	0.997	17.492	-6.320
23	1.045	14.708	2.480	57	1.011	16.014	-5.052
24	1.039	7.852	3.090	58	1.050	20.336	-7.000
25	1.060	9.698	2.240	59	1.063	22.564	-5.600
26	1.056	8.199	1.390	60	1.030	16.453	-5.400
27	1.043	6.314	2.810	61	1.025	20.788	-8.000
28	1.052	11.333	2.060	62	1.010	15.904	-5.000
29	1.051	13.970	2.840	63	1.000	18.347	-10.000
30	1.054	6.069	0.000	64	1.016	4.862	-13.500
31	1.057	8.630	0.000	65	1.011	0.000	-35.914
32	1.051	10.956	0.000	66	1.000	46.024	-17.850
33	1.056	7.473	1.120	67	1.000	39.785	-10.000
34	1.065	2.537	0.000	68	1.000	45.530	-40.000

Table 4.3. Pre-disturbance conditions for voltage, phase and net power injections for the IEEE 68-bus system.

Bus from	Bus to	Line Power Transfer (p.u.)	Bus from	Bus to	Line Power Transfer (p.u.)
1	2	-0.84	26	29	-1.75
1	30	1.51	28	29	-3.33
2	3	3.86	29	61	-7.95
2	25	-2.20	9	30	-3.68
2	53	-2.50	9	36	3.25
3	4	1.03	9	36	3.25
3	18	-0.41	36	37	24.91
4	5	-1.30	34	36	6.08
4	14	-2.68	35	34	0.00
5	6	-4.68	33	34	6.11
5	8	3.38	32	33	6.74
6	7	4.40	30	31	-2.66
6	11	-3.64	30	32	-3.23
6	54	-5.45	1	31	-2.38
7	8	2.05	31	38	-0.05
8	9	0.20	33	38	-0.52
9	30	-3.68	38	46	-0.58
10	11	3.66	46	49	-2.09
10	13	2.84	1	47	-0.84
10	55	-6.50	47	48	-1.44
12	11	-0.01	47	48	-1.44
12	13	-0.08	48	40	-5.30
13	14	2.75	35	45	0.00
14	15	0.06	37	43	0.54
15	16	-3.14	43	44	0.54
16	17	2.26	44	45	-2.49
16	19	-4.49	39	44	-0.35
16	21	-3.61	39	45	-2.32
16	24	-0.59	45	51	-6.90
17	18	1.99	50	52	-11.39
17	27	0.26	50	51	10.39
19	20	1.78	49	52	-3.74
19	56	-6.29	52	42	-0.10
20	57	-5.03	42	41	-1.60
21	22	-6.36	41	40	6.24
22	23	0.61	31	62	-5.00
22	58	-7.00	32	63	-10.00
23	24	3.71	36	64	-13.50
23	59	-5.59	37	65	-35.91
25	26	0.91	41	66	-17.85
25	60	-5.38	42	67	-10.00
26	27	2.53	52	68	-40.00
26	28	-1.26	1	27	0.02

Table 4.4. Pre-disturbance power transfers for the IEEE 68-bus system.

Chapter 5

Primary frequency regulation in power networks with ancillary service from load-side participation

This chapter considers the problem of designing distributed generation and demand control schemes that provide ancillary service in primary frequency regulation in power networks such that stability and fairness in the power allocation can be guaranteed. It is desirable in such schemes that load side participation in frequency control is activated only when the frequency exceeds prescribed thresholds so as to avoid frequent intervention in the operation of loads. This, however, leads to nonlinear control schemes with vector fields with discontinuous derivatives. In this chapter, we investigate how stability and optimality may be ensured when such dynamics are considered. We show that subgradient methods can be used to derive decentralised conditions that ensure optimality of the equilibrium points. Decentralised conditions for asymptotic stability, based on the analysis carried in Chapter 4, that are valid in this context are also provided. We illustrate our results with simulations on the IEEE 68 bus system, where it is demonstrated that the inclusion of controllable loads offers improved transient behaviour and that an optimal power allocation among controllable loads is achieved.

5.1 Introduction

We have discussed in the previous chapters that the increasing penetration of renewable sources of energy in the power grid is expected to cause fast changes in generation and therefore power imbalances which cannot be counter-balanced by conventional means of generation, due to their slow response. One potential solution to this problem comes from load participation that will provide fast response to these changes. However, if controllable appliances are to be used as a means of frequency regulation, it is imperative that a fair power allocation between them is achieved. This issue has been noted in literature, with studies such as [61] for primary control and [82, 119] for secondary control attempting to resolve it by providing conditions such that system equilibria coincide with the optimal solutions of an appropriately constructed optimisation problem that ensures a fair power allocation. As we have seen in Section 2.5.1, fairness is guaranteed in a distributed way within primary control by using frequency as a synchronisation signal.

Motivation: Controllable loads may provide ancillary services in the primary frequency control timeframe, responding only to frequency deviations beyond some threshold value in order to prevent unnecessary adjustments for small deviations. Hence, loads will support the power network with users comfort being rarely affected. Such behaviour is in line with the study of [40], that proposes a scheme where refrigerators switch on only when frequency exceeds some temperature dependent threshold. Similar models have been frequently used in literature with studies such as [120] and [121] using governor dead-bands to describe generation dynamics. Such dynamics contain vector fields with discontinuous derivatives, whose study is evidently highly relevant when controllable loads are considered.

Contribution and structure: In this chapter we show that subgradient methods can be used to derive decentralised conditions through which optimality in the power allocation can be deduced for frequency control schemes with non-differentiable vector fields. Furthermore, we discuss how the passivity based stability results, presented in Chapter 4, are still valid in this context and provide various realistic examples that fit within the framework considered. This allows to provide stability and optimality guarantees for broad classes of systems with realistic dynamics that are expected to be of practical importance when controllable loads assist in primary frequency control.

This chapter is structured as follows. In Section 5.2 we present the power network model, the classes of generation and controllable demand dynamics and the optimisation problem to be considered. Section 5.3 contains the main stability conditions. Section 5.4 includes our main results and in Section 5.5 we discuss how they apply to specific dynamics for controllable demand. In Section 5.6, we demonstrate our results through a simulation on the IEEE 68 bus system. Finally, conclusions are drawn in Section 5.7.

Note that the network model and main stability assumptions presented in Sections 5.2 and 5.3 follow from the analysis presented in Chapter 4. They are incorporated in this chapter for completeness and to justify the discussion and results presented in Section 5.5. Moreover, note that all notation used within this chapter follows from that introduced in Section 3.1.

5.2 Problem formulation

5.2.1 Generation and load dynamics

We shall represent frequency-dependent generation and demand dynamics using the class of systems introduced in this section. Appropriate conditions will be then imposed on those, in the sections that follow, to ensure stability and optimality of the equilibrium points when the dynamics are interconnected within a power system.

We consider dynamical systems with input $u(t) \in \mathbb{R}$, state $x(t) \in \mathbb{R}^m$ (for any $m \in \mathbb{N}$), and output $y(t) \in \mathbb{R}$ with a state space realization of the form

$$\begin{aligned} \dot{x} &= f(x, u), \\ y &= g(x, u), \end{aligned} \tag{5.1}$$

where $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ and $g : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous. It is assumed that the system (5.1) is such that given any constant input $u(t) \equiv \bar{u} \in \mathbb{R}$, there exists a unique[1] locally asymptotically stable equilibrium point $\bar{x} \in \mathbb{R}^m$, i.e. $f(\bar{x}, \bar{u}) = 0$. The region of attraction of $\bar{x}$ is denoted by X_0. We define the static input-state characteristic map by the function $k_x : \mathbb{R} \to \mathbb{R}^m$,

$$k_x(\bar{u}) := \bar{x}.$$

Based on this, we can also define the static input-output characteristic map k_y :

[1]This assumption can be relaxed to having isolated equilibrium points and is used here for simplicity.

$\mathbb{R} \to \mathbb{R}$,

$$k_y(\bar{u}) := g(k_x(\bar{u}), \bar{u}). \tag{5.2}$$

Remark 5.1 *Note that the local Lipschitz continuity condition on f allows discontinuities in the derivative of f, which is a property that will be considered within the chapter.*

5.2.2 Network model

We shall study the power network model presented in Chapter 2 which we describe below again for convenience. The power network is described by a connected graph (N, E) where $N = \{1, 2, \dots, |N|\}$ is the set of buses and $E \subseteq N \times N$ the set of transmission lines connecting the buses. It is assumed that the network consists of generation and load buses. Their main difference is that generation buses have non-zero generation inertia and nontrivial generation dynamics in contrast with load buses. Let $G = \{1, 2, \dots, |G|\}$ and $L = \{|G| + 1, \dots, |N|\}$ be the sets of generation and load buses such that $|G| + |L| = |N|$. Furthermore, we use (i, j) to denote the link connecting buses i and j and assume that the graph (N, E) is directed with arbitrary direction, so that if $(i, j) \in E$ then $(j, i) \notin E$. For each $j \in N$, we use $i : i \to j$ and $k : j \to k$ to denote the sets of buses that are predecessors and successors of bus j respectively. Note that any change in the graph ordering will not affect the form of the dynamics in (5.3)–(5.4) below and that all our results are independent of the choice of direction. The following assumptions are made for the network:

1) Bus voltage magnitudes are $|V_j| = 1$ p.u. for all $j \in N$.
2) Lines $(i, j) \in E$ are lossless and characterised by their susceptances $B_{ij} = B_{ji} > 0$.
3) Reactive power flows do not affect bus voltage phase angles and frequencies.

We use swing equations to describe the rate of change of frequency at generation buses, while power must be conserved at each of the load buses. This motivates the following system dynamics (e.g. [113]),

$$\dot{\eta}_{ij} = \omega_i - \omega_j,\ (i, j) \in E, \tag{5.3a}$$

$$M_j \dot{\omega}_j = -p_j^L + p_j^M - (d_j^c + d_j^u) - \sum_{k:j \to k} p_{jk} + \sum_{i:i \to j} p_{ij},\ j \in G, \tag{5.3b}$$

ω_j	frequency at bus j
η_{ij}	power angle difference between bus i and bus j
p_j^M	mechanical power injection at bus j
d_j^c	controllable load at bus j
d_j^u	uncontrollable frequency dependent load at bus j
p_{ij}	power transfer from bus i to bus j
B_{ij}	line susceptance between bus i and bus j
p_j^L	step change in uncontrollable demand at bus j
$x^{M,j}$	internal states of generation dynamics at bus j
$x^{c,j}$	internal states of controllable load dynamics at bus j
$x^{u,j}$	internal states of uncontrollable frequency dependent load dynamics at bus j

Table 5.1. Notation used in the system model (5.3)–(5.4). Note that variables ω_j, p_j^M, d_j^c, d_j^u, p_j^L denote deviations from corresponding nominal values and that by internal states we refer to the states in the state space representation of the differential equations representing the dynamics (details can be found in Sections 5.2.1 and 5.2.2).

$$0 = -p_j^L - (d_j^c + d_j^u) - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij},\ j \in L, \tag{5.3c}$$

$$p_{ij} = B_{ij} \sin \eta_{ij} - p_{ij}^{nom},\ (i,j) \in E. \tag{5.3d}$$

In system (5.3) the time-dependent variables p_j^M, ω_j and d_j^c represent, respectively, deviations from a nominal value[2] of the mechanical power injection to the generator bus j, and the frequency and controllable load present at any bus j. The quantity d_j^u is also a time-dependent variable that represents the uncontrollable frequency-dependent load and generation damping present at bus j. Furthermore, the quantities η_{ij} and p_{ij} are time-dependent variables that represent, respectively, the power angle difference, and the deviation from the nominal value, p_{ij}^{nom}, of the power transmitted from bus i to bus j. The constant $M_j > 0$ denotes the generator inertia. We study the response of system (5.3) at a step change in the uncontrollable demand p_j^L at each bus j.

To investigate distributed control schemes for generation and controllable loads, we determine each of the scalar variables p_j^M, d_j^c, and d_j^u as outputs from independent systems of the form described in Section 5.2.1 with inputs given by the negative of

[2]A nominal value is defined as an equilibrium of (5.3) with frequency equal to 50 Hz (or 60 Hz).

the local frequency,

$$\begin{aligned} \dot{x}^{M,j} &= f^{M,j}(x^{M,j}, -\omega_j), \\ p_j^M &= g^{M,j}(x^{M,j}, -\omega_j), \end{aligned} \qquad j \in G, \tag{5.4a}$$

$$\begin{aligned} \dot{x}^{c,j} &= f^{c,j}(x^{c,j}, -\omega_j), \\ d_j^c &= g^{c,j}(x^{c,j}, -\omega_j), \end{aligned} \qquad j \in N, \tag{5.4b}$$

$$\begin{aligned} \dot{x}^{u,j} &= f^{u,j}(x^{u,j}, -\omega_j), \\ d_j^u &= g^{u,j}(x^{u,j}, -\omega_j), \end{aligned} \qquad j \in N. \tag{5.4c}$$

For notation, we collect[3] the variables in (5.4) into the vectors $x^M = [x^{M,j}]_{j\in G}$, $x^c = [x^{c,j}]_{j\in N}$, and $x^u = [x^{u,j}]_{j\in N}$. These quantities represent the internal states of the dynamical systems used to update the desired outputs p_j^M, d_j^c, and d_j^u. Since p_j^M and d_j^c are controllable, we have freedom to design certain properties of the dynamics in (5.4a) and (5.4b). By contrast, the dynamics in (5.4c) cannot be controlled since d_j^u represents uncontrollable load and generation damping. All systems in (5.4) can be of arbitrary (heterogeneous) dimensions and can have any form so long as they fit within the framework introduced in Section 5.2.1 and satisfy the assumptions that will be introduced below.

5.2.3 Optimal supply and load control

We aim to investigate how to adjust generation and controllable demand such that the step change p^L in demand is met and at the same time the total cost that comes from the extra power generated and the disutility of loads is minimised. In order to achieve this, we introduce the following optimisation problem, which we call the optimal supply and load control problem (OSLC).

Suppose that deviations p_j^M and d_j^c in generation and controllable load result in costs $C_j(p_j^M)$ and $C_{dj}(d_j^c)$ respectively. Moreover, any change in frequency alters frequency dependent uncontrollable demand and therefore also incurs some additional cost. This cost is represented by the negative integral of the function $k_{d_j^u}$ which is explicitly determined by the dynamics in (5.4c). The total cost within OSLC is the sum of all the above costs. The problem is to choose the vectors p^M, d^c, and d^u such that this total cost is minimised when simultaneously power balance is achieved,

[3] Each local variable (e.g. $x^{M,j}$) is a vector with multiple components.

and physical saturation constraints are satisfied.

$$\begin{aligned}
&\underline{\text{OSLC:}}\\
&\min_{p^M, d^c, d^u} \sum_{j \in G} C_j(p_j^M) + \sum_{j \in N} \Big(C_{dj}(d_j^c) - \int_0^{d_j^u} k_{d_j^u}^{-1}(z)\, dz \Big)\\
&\text{subject to } \sum_{j \in G} p_j^M = \sum_{j \in N} (d_j^c + d_j^u + p_j^L),\\
&p_j^{M,\min} \leq p_j^M \leq p_j^{M,\max},\ \forall j \in G,\\
&d_j^{c,\min} \leq d_j^c \leq d_j^{c,\max},\ \forall j \in N,
\end{aligned} \tag{5.5}$$

where $p_j^{M,\min}, p_j^{M,\max}, d_j^{c,\min}$, and $d_j^{c,\max}$ are the bounds for generation and controllable demand respectively at bus j. The equality constraint in (5.5) represents conservation of power by specifying that all the extra frequency-independent load is matched by the total additional generation plus all the deviations in frequency-dependent loads.

Remark 5.2 *The fact that no integral action is present in primary control leads to a non-zero steady state frequency deviation. This can be used as a synchronisation signal through which an optimisation interpretation can be provided for the power generated/consumed, without requiring additional information exchange.*

Remark 5.3 *It is important to note that the only design variables within* (5.5) *are p^M and d^c which represent generation and controllable demand respectively. In contrast, the variable d^u can be controlled only indirectly by effecting changes in the system frequencies (uncontrollable frequency-dependent demand). Therefore, we aim to specify properties for p^M and d^c that will ensure their convergence to an equilibrium point where optimality in* (5.5) *is ensured.*

The following assumption is imposed on the cost functions in (5.5).

Assumption 5.1 *The cost functions C_j and C_{dj} are continuous and strictly convex. Furthermore, they are continuously differentiable except on respective sets Λ_j and Λ_{dj} of isolated points. Moreover, the first derivative of $k_{d_j^u}^{-1}(z)$ is non-positive for all $z \in \mathbb{R}$.*

Assumption 5.1 allows for optimisation guarantees for cost functions that are nondifferentiable at a discrete set of points. This permits classes of hybrid cost functions and more involved control dynamics. To overcome the issue of nondifferentiability within the optimality proof (Theorem 5.1), we will consider the notions of subgradient and subdifferential (e.g. [105]), defined below.

Definition 5.1 *Given a convex function $f : I \to \mathbb{R}$, a subgradient of f at a point $x \in I \subseteq \mathbb{R}$ is any $v \in \mathbb{R}$ such that $f(y) - f(x) \geq v(y-x)$ for all $y \in I$. The set of all subgradients of f at x is called the subdifferential of f at x and is denoted by $\partial f(x)$.*

For the optimality analysis, we shall make use of C'_j and C'_{dj} and their respective inverses. Since, however, these derivatives are allowed to be discontinuous, their inverses are not well defined at points of discontinuity. We therefore need to define the following functions that are equal to the inverse map of the subdifferential of C'_j and C'_{dj} respectively, and can be seen as generalized inverses of C'_j and C'_{dj}. In particular, we define

$$\begin{aligned} D_j^{-1}(x) &:= \begin{cases} (C'_j)^{-1}(x), & x \in C'_j(\mathbb{R} \setminus \Lambda_j), \\ \gamma, & x \in [C'_j(\gamma-), C'_j(\gamma+)], \gamma \in \Lambda_j, \end{cases} \\ D_{dj}^{-1}(x) &:= \begin{cases} (C'_{dj})^{-1}(x), & x \in C'_{dj}(\mathbb{R} \setminus \Lambda_{dj}), \\ \gamma, & x \in [C'_{dj}(\gamma-), C'_{dj}(\gamma+)], \gamma \in \Lambda_{dj}, \end{cases} \end{aligned} \tag{5.6}$$

where the quantities $\gamma\pm$ respectively denote the limits $\lim_{\epsilon\downarrow 0}(\gamma \pm \epsilon)$.

The functions D_j^{-1} and D_{dj}^{-1} defined in (5.6) are therefore equal to C'^{-1}_j and C'^{-1}_{dj} respectively in the regime where C'_j and C'_{dj} are continuous, and otherwise remain constant. They are thus continuous non decreasing functions and are well defined at all points.

Note that the map of any point $x \in \mathbb{R}$ under D_j and D_{dj} is given by

$$\begin{aligned} D_j(x) &= \begin{cases} C'_j(x), & x \in \mathbb{R} \setminus \Lambda_j, \\ [C'_j(\gamma-), C'_j(\gamma+)], & x = \gamma \in \Lambda_j, \end{cases} \\ D_{dj}(x) &= \begin{cases} C'_{dj}(x), & x \in \mathbb{R} \setminus \Lambda_{dj}, \\ [C'_{dj}(\gamma-), C'_{dj}(\gamma+)], & x = \gamma \in \Lambda_{dj}. \end{cases} \end{aligned} \tag{5.7}$$

It therefore follows from (5.7), the fact that C'_j and C'_{dj} are strictly increasing, and Definition 5.1 that, for all $x \in \mathbb{R}$,

$$D_j(x) = \partial C_j(x) \text{ and } D_{dj}(x) = \partial C_{dj}(x). \tag{5.8}$$

i.e. $D_j(x)$, $D_{dj}(x)$ are equal to the subdifferential of C_j, C_{dj} at x respectively.

5.2.4 Equilibrium analysis

We now quantify what is meant by an equilibrium of the interconnected system (5.3)–(5.4).

Definition 5.2 *The constants* $(\eta^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*})$ *define an equilibrium of the system* (5.3)–(5.4) *if all time derivatives of* (5.3)–(5.4) *are zero.*

Note that the static input-output characteristic maps $k_{p_j^M}$, $k_{d_j^c}$ and $k_{d_j^u}$ as defined in Section 5.2.1, suffice to completely describe the effect of the dynamics (5.4) on the equilibrium behaviour of the power system (5.3). Furthermore, we assume that there exists some equilibrium of (5.3)–(5.4) as defined in Definition 5.2.

We now impose an assumption on the equilibrium, which can be interpreted as a security constraint for the power flows generated.

Assumption 5.2 $|\eta_{ij}^*| < \frac{\pi}{2}$ *for all* $(i,j) \in E$.

5.3 Combined passive dynamics for generation and demand

In this section we state various assumptions, reproduced from Chapter 4, that are required for the convergence result presented in Section 5.4.

It is useful to define the net supply variables below that result from the outputs of (5.4),

$$s_j = p_j^M - (d_j^c + d_j^u),\ j \in G, \tag{5.9a}$$

$$s_j = -(d_j^c + d_j^u),\ j \in L. \tag{5.9b}$$

Correspondingly, their equilibrium values can be written as $s_j^* = p_j^{M,*} - (d_j^{c,*} + d_j^{u,*})$ and $s_j^* = -(d_j^{c,*} + d_j^{u,*})$ for $j \in G$ and $j \in L$ respectively.

Since the variables in (5.9) are the aggregation of the variables described in (5.4), their dynamics can be viewed as outputs from these combined dynamical systems with inputs $(-\omega_j)$.

The following notion of passivity for systems of the form (5.1) will be used to characterise the dynamics of the supply variables defined in (5.9) and is required for our stability results.

Definition 5.3 *The system* (5.1) *is said to be locally input strictly passive about the constant input values $\bar{u}$ and the constant state values $\bar{x}$ if there exist open neighbourhoods U of $\bar{u}$ and X of $\bar{x}$ and a continuously differentiable, positive semidefinite function $V(x)$ (the storage function) such that, for all $u \in U$ and all $x \in X$, $\dot{V}(x) \leq (u - \bar{u})^T(y - \bar{y}) - \phi(u - \bar{u})$, where ϕ is a positive definite function and $\bar{y} = k_y(\bar{u})$.*

We assume that this condition is satisfied about equilibrium by the supply dynamics (5.9). Note that this is a decentralised condition.

Assumption 5.3 *Each of the systems defined in* (5.4) *with inputs* $(-\omega_j)$ *and outputs given by* (5.9a) *and* (5.9b) *respectively are locally input strictly passive about their equilibrium values* $(-\omega_j^*)$ *and* $(x^{M,j,*}, x^{c,j,*}, x^{u,j,*})$, *in the sense described in Definition 5.3.*

Remark 5.4 *Note that for linear systems Assumption 5.3 can be checked from the strict positive realness of the corresponding transfer function or numerically using the KYP Lemma (e.g. [49]). This can also be easily checked for static nonlinearities and classes of higher order nonlinear systems as discussed in Section 5.5.*

Assumption 5.4 *The storage functions in Assumption 5.3 have strict local minima at the points* $(x^{M,j,*}, x^{c,j,*}, x^{u,j,*})$ *and* $(x^{c,j,*}, x^{u,j,*})$ *respectively.*

Remark 5.5 *In practice, Assumption 5.4 is often satisfied. For instance, for a linear system the KYP Lemma generates a storage function satisfying Assumption 5.4 whenever this is controllable and observable.*

The following condition is also required in order for stability and fairness to be guaranteed. Within it, we denote $\omega^G = [\omega_j]_{j \in G}$ and $\omega^L = [\omega_j]_{j \in L}$.

Assumption 5.5 *There exists an open neighbourhood T of $(\eta^*, \omega^{G,*}, x^{M,*}, x^{c,*}, x^{u,*})$ and a locally Lipschitz map f^L such that when $(\eta, \omega^G, x^M, x^c, x^u) \in T$ it holds that $\omega^L = f^L(\eta, \omega^G, x^M, x^c, x^u)$.*

Remark 5.6 *Assumption 5.5 is a technical assumption that is required in order for the system* (5.3)–(5.4) *to have a locally well-defined state space realization.*

5.4 Main results

This section contains the main results of this chapter, with the proofs provided in the appendix. The first result provides sufficient conditions for equilibrium points to solve the OSLC problem (5.5) and ensure a fair power allocation. Our second result guarantees convergence to optimality of all solutions starting in the vicinity of an equilibrium of the system (5.3)–(5.4) for which the assumptions stated are satisfied.

Theorem 5.1 *Suppose that Assumption 5.1 is satisfied. If the control dynamics in* (5.4a) *and* (5.4b) *are chosen such that*

$$\begin{aligned} k_{p_j^M}(-\omega_j^*) &= [D_j^{-1}(-\omega_j^*)]_{p_j^{M,\min}}^{p_j^{M,\max}}, \\ k_{d_j^c}(-\omega_j^*) &= [D_{dj}^{-1}(\omega_j^*)]_{d_j^{c,\min}}^{d_j^{c,\max}}, \end{aligned} \tag{5.10}$$

then the values $p^{M,*}$, $d^{c,*}$, and $d^{u,*}$ are optimal for the OSLC problem (5.5).

Theorem 5.2 *Consider equilibria of* (5.3)–(5.4) *with respect to which Assumptions 5.1–5.5 are all satisfied. If the control dynamics in* (5.4a) *and* (5.4b) *are chosen such that*

$$\begin{aligned} k_{p_j^M}(\bar{u}) &= [D_j^{-1}(\bar{u})]_{p_j^{M,\min}}^{p_j^{M,\max}} \\ k_{d_j^c}(\bar{u}) &= [D_j^{-1}(-\bar{u})]_{d_j^{c,\min}}^{d_j^{c,\max}} \end{aligned} \tag{5.11}$$

hold for all $\bar{u} \in \mathbb{R}$, then there exists an open neighbourhood of initial conditions about any such equilibrium such that the solutions of (5.3)–(5.4) *are guaranteed to converge to a global minimum of the OSLC problem* (5.5).

Remark 5.7 *Theorem 5.1 extends the optimality result in Theorem 4.2 from Chapter 4 to the case where the vector field has discontinuous derivatives, by replacing the derivative of the cost function with its subdifferential. Theorem 5.2 illustrates that the stability conditions established in Chapter 4 are still valid in this context. These results are important in practical implementations as discussed in Section 5.5.*

5.5 Discussion

In this section we highlight the importance of our contribution by discussing examples that fit within the framework presented in this chapter.

An important class of droop control schemes used in practice incorporates a deadband which prevents unneeded adjustments for small variations in frequency about its nominal value. An example of this is shown in Fig. 5.1, with controllable load dynamics that respond to frequency only if it overpasses some threshold value. For such systems, a minimum frequency deviation ω_j^0 is required to trigger a frequency-dependent change. The system then reaches its physical limits at a higher frequency deviation ω_j^1.

This example can be generalized to set-point inputs p_j^c with the following dynamics,

$$p_j^c(\omega_j) = \begin{cases} p_j^{c,max}, \quad \omega_j < \omega_j^{th,1} \\ k_{n,j}(\omega_j - \omega_j^{th,n}) + c_{n,j}, \ \omega_j^{th,n} \leq \omega_j < \omega_j^{th,n+1} \\ p_j^{c,min}, \quad \omega_j > \omega_j^{th,N_j} \end{cases} \tag{5.12}$$

for $n = \{1, 2, \ldots, N_j - 1\}$, where $k_{n,j} \leq 0$, $c_{n,j} = k_{n,j}(\omega_j^{th,n} - \omega_j^{th,n-1}) + c_{n-1,j}$, $c_{1,j} = p_j^{c,max}$ and $p_j^{c,min}$,$p_j^{c,max} \in \mathbb{R}$ satisfy $-\infty < p_j^{c,min} \leq p_j^{c,max} < \infty$. An example of such dynamics with $N_j = 6$ is shown in Figure 5.2. Such systems permit different droop gain for different frequency regimes. Controllable loads with set points as in (5.12) offer a small contribution when frequency deviation exceeds some threshold to prevent a potential urgency and a bigger contribution when frequency passes a different threshold where the urgency occurs. Note that if (5.12) is expressed as $p_j^c = h_j(-\omega_j)$, then $h_j^{-1}(.)$ is a strictly increasing function in the domain $(-\omega_j^{th,N_j}, -\omega_j^{th,1})$. This is compatible with the optimisation interpretation given in Theorem 5.1, since function $D_j(.)$ in (5.6) is strictly increasing due to the convexity of $C_j(.)$.

Our stability and optimality results apply to systems described by:

$$s_j = K_j(p_j^c(\omega_j)) \tag{5.13a}$$

$$d_j^u = \beta_j \omega_j \tag{5.13b}$$

where $K_j(.)$ is any continuous increasing function, $\beta_j > 0$ is a constant corresponding to system damping, and function $p_j^c(.)$ is as described in (5.12). For such systems, it is easy to verify that all conditions of Theorems 5.1 and 5.2 are satisfied.

Before providing some more involved examples, it would be useful to consider Proposition 4.1 from Chapter 4, which is re-stated[4] below.

[4]Proposition 4.1 is stated as a gain condition from $(\omega_j - \omega_j^*)$ to $(p_j^M - p_j^{M,*})$, but the extension here is trivial.

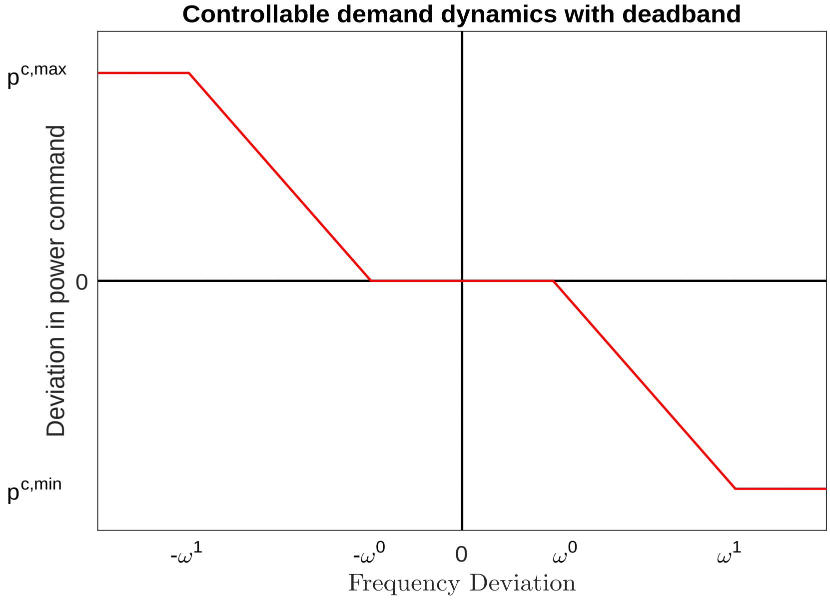

Figure 5.1. Power supply set point against frequency deviations for a system with dead-band and saturation bounds.

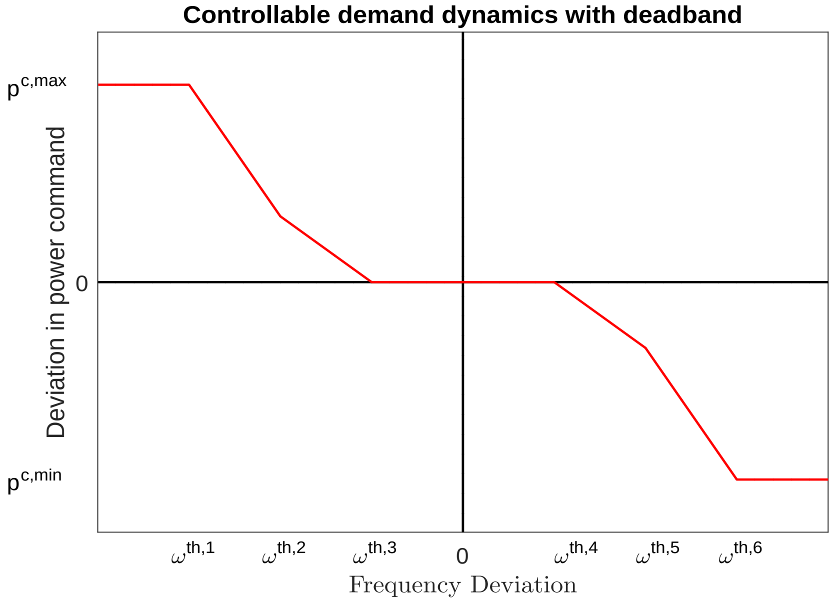

Figure 5.2. Power supply set point against frequency deviations as described by (5.12) with 6 threshold frequencies, dead-band and saturation bounds.

Proposition 5.1 *Let equation* (5.13b) *hold and consider any dynamics from* $-\omega_j$ *to* $(p_j^M - d_j^c)$ *of the form* (5.4a). *Given any equilibrium, if the* $\mathcal{L}_2$*-gain from* $(\omega_j - \omega_j^*)$ *to* $((p_j^M - d_j^c) - (p_j^{M,*} - d_j^{c,*}))$ *is strictly less than* β_j*, then the system with input* $-\omega_j$ *and output* s_j *is input strictly passive about the equilibrium considered.*

Using Proposition 5.1, we can show how general higher order non-linear systems with non-C^1 vector fields may fit within our framework.

To illustrate the applicability of our approach we consider below a 5th order model for generation dynamics. This is a realistic model used by the power system toolbox [115] to describe turbine/governor dynamics in the Northeast Power Coordinating Council (NPCC) network. The dynamics are described by the transfer function below, from negative frequency to generation output,

$$G_j(s) = K_j \frac{(1 + sT_{s,j})(1 + sT_{3,j})}{(1 + sT_{c,j})(1 + sT_{4,j})(1 + sT_{5,j})} \tag{5.14}$$

where K_j and $T_{s,j}, T_{3,j}, T_{c,j}, T_{4,j}, T_{5,j}$ are the droop coefficient and time-constants respectively. If the input to system (5.14) is signal $p_j^c(\omega_j)$ in (5.12) then droop control becomes active only after some frequency deviation has been exceeded, thus providing an ancillary service for large enough frequency deviations. Moreover, droop control schemes as in (5.12) may allow for more cost effective generation if the values of $k_{n,j}$ are properly selected. Our results allow optimality to be deduced for such a system, by making use of Theorem 5.1.

Furthermore, noting that system (5.14) has an $\mathcal{L}_2$-gain of K_j, a sufficient stability condition for generation dynamics as in (5.14), with input given by (5.12) and damping as in (5.13b) is provided in the corollary below, which is proved in the Appendix.

Corollary 5.1 *Let equation* (5.13b) *hold and consider the generation dynamics described by* (5.14) *with input given by* (5.12). *Then the system with input* $-\omega_j$ *and output* $p_j^M - d_j^u$ *is input strictly passive about any equilibrium, if* (5.12) *satisfies* $\max_n |k_{n,j}| < \beta_j / K_j$.

It should also be noted that versions of the KYP Lemma ([122], Lemma 7.3) can be used in this case to deduce the storage function of the system.

Apart from their use in conventional droop control schemes at generating units, deadbands in the control policy can also be important in highly distributed schemes where smart appliances or loads act as ancillary services when large deviations in frequency occur. The results in this chapter show that stability and optimality can be guaranteed in a distributed way despite the presence of such nonlinearities in the feedback policy.

5.6 Simulations

Our stability and optimality results are illustrated within this section through applications on the IEEE New York / New England 68 bus interconnection system [116], simulated using the Power System Toolbox [115]. The toolbox uses a more detailed and realistic model than our analytic one, including line resistances, a DC12 exciter model, power system stabilizer (PSS), and a subtransient reactance generator model.

The test system consists of 52 load buses serving different types of loads including constant active and reactive loads and 16 generation buses. The overall system has a total real power of 16.41GW. For our simulation, we added five loads on units 2, 8, 9, 17, and 25, each having a step increase of magnitude 1 p.u. (base 100MVA) at $t = 1$ second. Controllable demand was included on all loads buses and loads were controlled every 10ms.

Controllable loads were designed to remain constant for frequency deviations less than $\omega^0 = 0.1$Hz. The disutility function and dynamics for controllable loads d_j^c are respectively given by

$$C_{dj}(d_j^c) = \begin{cases} \frac{1}{2}\alpha_j (d_j^c)^2 + \omega^0 d_j^c, & d_j^c \geq 0 \\ \\ \frac{1}{2}\alpha_j (d_j^c)^2 - \omega^0 d_j^c, & d_j^c < 0 \end{cases} \quad j \in N, \tag{5.15}$$

$$d_j^c = (C'_{dj})^{-1}(\omega_j) = \begin{cases} \frac{1}{\alpha_j}(\omega_j - \omega^0), \ \omega_j > \omega^0 \\ 0, \ -\omega^0 \leq \omega_j \leq \omega^0, \\ \frac{1}{\alpha_j}(\omega_j + \omega^0), \ \omega_j < -\omega^0 \end{cases} \quad j \in N, \tag{5.16}$$

where the selected values for cost coefficients were $\alpha_j = 2$ for load buses $1 - 10$ and $\alpha_j = 4$ for the rest. We shall refer to the resulting dynamics as Static OSLC since its equilibria solve the OSLC problem. This scheme satisfies Assumption 5.3 in the presence of arbitrarily small frequency damping, and is thus included within our framework.

The system was tested for two cases: i) With no controllable loads, ii) With controllable loads satisfying the Static OSLC scheme. The frequency response under those two cases for bus 63 is shown in Fig. 5.3. From Fig. 5.3, we observe that the steady state frequency deviation is smaller in the presence of controllable loads and also that the system converges faster to that value. The fact that in both cases

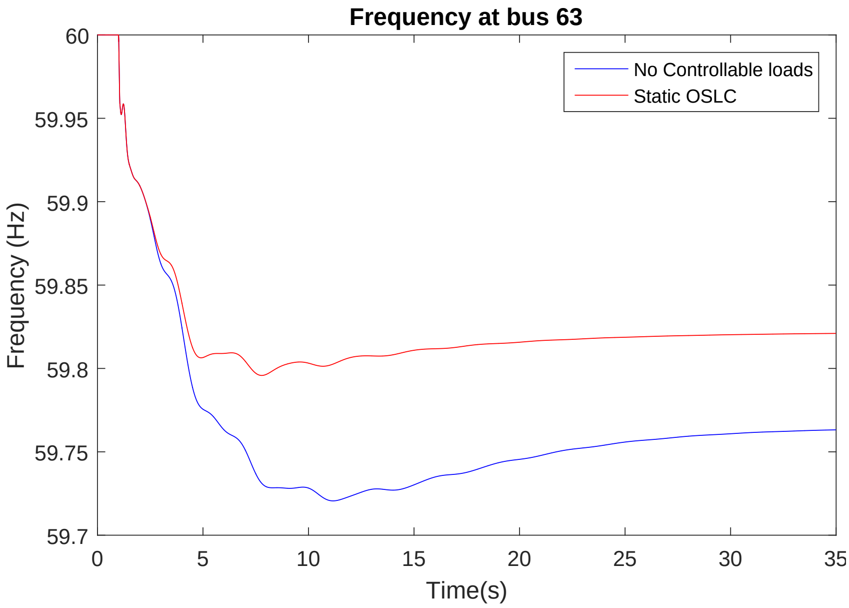

Figure 5.3. Frequency at bus 63 for two cases: i) No controllable loads, ii) Static OSLC.

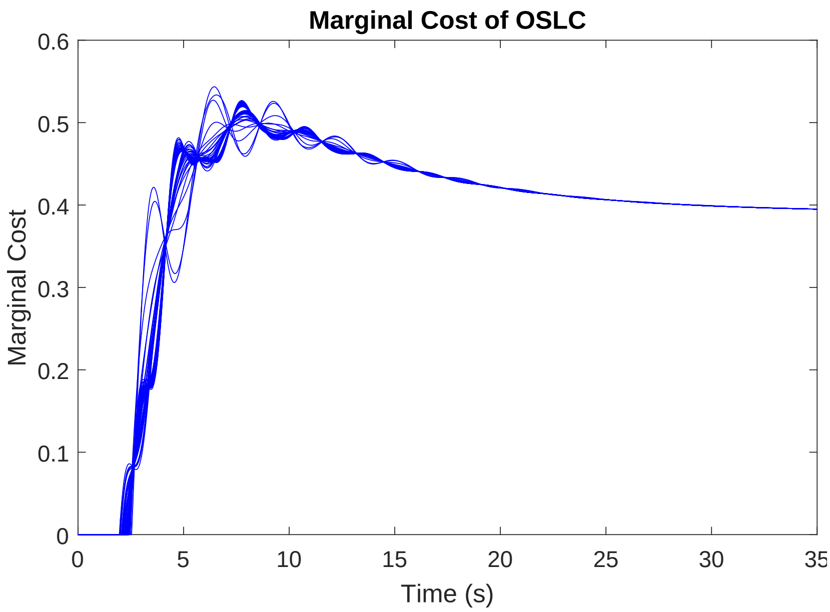

Figure 5.4. Marginal costs for controllable loads with non-equal cost coefficients for Static OSLC.

the frequency converges to an equilibrium value demonstrates the stability of the system. Furthermore, from Fig. 5.4, we observe that the marginal costs, defined as $-C'_{dj}(d^c_j)$, at each controlled load converge to the same value. This illustrates the optimality in the power allocation among loads, since the equality in marginal cost is necessary to solve (5.5) when the power generated/consumed does not saturate to its maximum/minimum value.

5.7 Conclusion

We have considered the problem of designing distributed schemes for primary frequency control in power networks involving systems with non-differentiable vector fields, such that stability and optimality of the power allocation can be guaranteed. In particular, we have shown that the use of subgradient methods allows to derive decentralised conditions for optimality and also discussed conditions through which stability can be deduced in this context. Our results have been illustrated with simulations on the IEEE 68 bus system.

Appendix

Within the proof of Theorem 5.1 we will make use of the following equilibrium equations for system (5.3)–(5.4), which follow from Definition 5.2.

$$0 = \omega_i^* - \omega_j^*,\ (i,j) \in E, \tag{5.17a}$$

$$0 = -p_j^L + p_j^{M,*} - (d_j^{c,*} + d_j^{u,*}) - \sum_{k:j\to k} p_{jk}^* + \sum_{i:i\to j} p_{ij}^*,\ j \in G, \tag{5.17b}$$

$$0 = -p_j^L - (d_j^{c,*} + d_j^{u,*}) - \sum_{k:j\to k} p_{jk}^* + \sum_{i:i\to j} p_{ij}^*,\ j \in L, \tag{5.17c}$$

$$p_j^{M,*} = k_{p_j^M}(-\omega_j^*),\ j \in G, \tag{5.17d}$$

$$d_j^{c,*} = k_{d_j^c}(-\omega_j^*),\ d_j^{u,*} = k_{d_j^u}(-\omega_j^*),\ j \in N. \tag{5.17e}$$

Proof of Theorem 5.1: The proof of this theorem is based on subgradient techniques [105, Section 23]. Firstly we note that strict convexity implies both that C_j' and C_{dj}' are strictly increasing on $\mathbb{R} \setminus \Lambda_j$ and $\mathbb{R} \setminus \Lambda_{dj}$ respectively, and that their jumps on Λ_j and Λ_{dj} are positive. Therefore, C_j' and C_{dj}' are invertible on $\mathbb{R} \setminus \Lambda_j$ and $\mathbb{R} \setminus \Lambda_{dj}$, and $C_j'(\gamma-) < C_j'(\gamma+)$ for all $\gamma \in \Lambda_j$ and $C_{dj}'(\gamma-) < C_{dj}'(\gamma+)$ for all $\gamma \in \Lambda_{dj}$. Moreover, continuous differentiability on the sets $\mathbb{R} \setminus \Lambda_j$ and $\mathbb{R} \setminus \Lambda_{dj}$ ensures that the relevant limits $\gamma-$ and $\gamma+$ here all exist. Consequently, the functions in (5.6) and hence the controls in (5.10) are well-defined. Moreover, Assumption 5.1 also ensures that the OSLC problem (5.5) is a convex optimisation problem. Thus, a point $(\bar{p}^M, \bar{d}^c, \bar{d}^u)$ is a global minimum for (5.5) if and only if it satisfies the subgradient KKT conditions,

$$-\nu - \lambda_j^+ + \lambda_j^- \in \partial C_j(\bar{p}_j^M),\ j \in G, \tag{5.18a}$$

$$\nu - \mu_j^+ + \mu_j^- \in \partial C_{dj}(\bar{d}_j^c),\ j \in N, \tag{5.18b}$$

$$k_{d_j^u}^{-1}(\bar{d}_j^u) = -\nu,\ j \in N, \tag{5.18c}$$

$$\sum_{j\in G} \bar{p}_j^M = \sum_{j\in N} (\bar{d}_j^c + \bar{d}_j^u + p_j^L), \tag{5.18d}$$

$$p_j^{M,\min} \le \bar{p}_j^M \le p_j^{M,\max},\ j \in G, \tag{5.18e}$$

$$d_j^{c,\min} \le \bar{d}_j^c \le d_j^{c,\max},\ j \in N, \tag{5.18f}$$

$$\lambda_j^+(\bar{p}_j^M - p_j^{M,\max}) = 0,\ \lambda_j^-(\bar{p}_j^M - p_j^{M,\min}) = 0,\ j \in G, \tag{5.18g}$$

$$\mu^+(\bar{d}_j^c - d_j^{c,\max}) = 0,\ \mu^-(\bar{d}_j^c - d_j^{c,\min}) = 0,\ j \in N, \tag{5.18h}$$

for some constants $\nu \in \mathbb{R}$ and $\lambda_j^+, \lambda_j^-, \mu_j^+, \mu_j^- \geq 0$. We will now show that these conditions are satisfied by the equilibrium values $(\bar{p}^M, \bar{d}^c, \bar{d}^u) = (p^{M,*}, d^{c,*}, d^{u,*})$ defined by equations (5.10), (5.17d) and (5.17e).

To deal with the inequality constrains and the discontinuities in the cost function derivatives, we define the following quantities

$$\begin{aligned} \omega_j^{M,\max} &:= \sup\{\omega_j \colon D_j^{-1}(-\omega_j) \geq p_j^{M,\max}\}, \\ \omega_j^{M,\min} &:= \inf\{\omega_j \colon D_j^{-1}(-\omega_j) \leq p_j^{M,\min}\}, \\ \omega_j^{c,\max} &:= \inf\{\omega_j \colon D_{dj}^{-1}(\omega_j) \geq d_j^{c,\max}\}, \\ \omega_j^{c,\min} &:= \sup\{\omega_j \colon D_{dj}^{-1}(\omega_j) \leq d_j^{c,\max}\}. \end{aligned} \tag{5.19}$$

Letting ω_0^* denote the common value of all ω_j^* due to (5.17a), we can also, in terms of these quantities, define the nonnegative constants

$$\lambda_j^+ := (\omega_j^{M,\max} - \omega_0^*)\, \mathbb{1}_{\{q \colon q \leq \omega_j^{M,\max}\}}(\omega_0^*),$$

$$\begin{aligned} \lambda_j^- &:= (\omega_0^* - \omega_j^{M,\min})\, \mathbb{1}_{\{q \colon q \geq \omega_j^{M,\min}\}}(\omega_0^*), \\ \mu_j^+ &:= (\omega_0^* - \omega_j^{c,\max})\, \mathbb{1}_{\{q \colon q \geq \omega_j^{c,\max}\}}(\omega_0^*), \\ \mu_j^- &:= (\omega_j^{c,\min} - \omega_0^*)\, \mathbb{1}_{\{q \colon q \leq \omega_j^{c,\min}\}}(\omega_0^*). \end{aligned}$$

Then, since $(D_j)^{-1}(-\omega_0^*) \geq p_j^{M,\max} \Leftrightarrow \omega_0^* \leq \omega_j^{M,\max}$, $(D_j)^{-1}(-\omega_0^*) \leq p_j^{M,\min} \Leftrightarrow \omega_0^* \geq \omega_j^{M,\min}$, $(D_{dj})^{-1}(\omega_0^*) \geq d_j^{c,\max} \Leftrightarrow \omega_0^* \geq \omega_j^{c,\max}$, and $(D_{dj})^{-1}(\omega_0^*) \leq d_j^{c,\min} \Leftrightarrow \omega_0^* \leq \omega_j^{c,\min}$, it follows by (5.17d), (5.17e) and (5.10) that the complementary slackness conditions (5.18g) and (5.18h) are satisfied.

Let $\nu = \omega_0^*$. Then, we deduce by (5.17e) that (5.18c) is trivially satisfied when $\bar{d}_j^u = d_j^{u,*}$. Furthermore, taking the preimages of $p_j^{M,*}$ and $d_j^{c,*}$ under D_j and D_{dj} respectively yields $-\nu - \lambda_j^+ + \lambda_j^- \in D_j(p_j^{M,*})$ and $\nu - \mu_j^+ + \mu_j^- \in D_{dj}(d_j^{c,*})$, hence we deduce by (5.8) that (5.18a) and (5.18b) are both satisfied by $\bar{p}_j^M = p_j^{M,*}$ and $\bar{d}_j^c = d_j^{c,*}$. Moreover, summing the equilibrium conditions (5.17b) over all $j \in G$ and (5.17c) over all $j \in L$ shows that (5.18d) holds. Finally, the saturation constraints in (5.10) verify (5.18e) and (5.18f).

Therefore, all of the subgradient KKT conditions (5.18a)–(5.18h) are satisfied, and so $(p^{M,*}, d^{c,*}, d^{u,*})$ defines a global optimum for the OSLC problem (5.5). ■

Proof of Theorem 5.2: The convergence part of the proof for Theorem 5.2 is identical to the proof of Theorem 4.1, noting that local Lipschitz continuity of the vector field is sufficient for the Lyapunov arguments used. Hence, if Assumptions 5.1–5.5

all hold and (5.11) is true, then all of the assumptions in Theorem 5.1 in this chapter and Theorem 4.1 are satisfied, and thus the result follows. ■

Proof of Corollary 5.1: The condition $\max_n |k_{n,j}| < \beta_j/K_j$ in (5.12) implies that the $\mathcal{L}_2$-gain from $(\omega_j - \omega_j^*)$ to $(p_j^c(\omega_j) - p_j^c(\omega_j^*))$ is less than β_j/K_j for any equilibrium $(\omega_j^*, p_j^c(\omega_j^*))$. Hence, since (5.14) has an $\mathcal{L}_2$-gain of K_j, it follows that the $\mathcal{L}_2$ gain from $(\omega_j - \omega_j^*)$ to $(p_j^M - p_j^{M,*})$ is less than β_j for any feasible equilibrium pair $(\omega_j^*, p_j^{M,*})$. Therefore, the condition in Proposition 5.1 is satisfied and the system from $(-\omega_j)$ to $(p_j^M - d_j^u)$ is input strictly passive. ■

Part II

Secondary frequency regulation with load-side participation

Contribution of this part

In this part we study the behaviour of power systems at the presence of controllable loads and develop distributed feedback schemes for secondary frequency regulation. The additional complication compared to Part I is the requirement for frequency to return to its nominal value. This makes several approaches adopted in Part I unsuitable to be used in this context. The main contributions of this part, with pointers to the main assumptions and theorems, are summarised below.

In Chapter 6 we consider the stability and optimality of power networks within the secondary frequency control timeframe. We present a systematic method for the design of distributed schemes for generation and demand such that stability and fairness in power allocation are ensured. A locally communicated power command signal is used as a synchronising variable with dynamics that ensured that frequency will be equal to the nominal at steady state. Under this scheme, we provide a decentralised dissipativity condition on the generation and controllable demand aggregate dynamics in each bus such that convergence to equilibrium is guaranteed. The main stability condition in this chapter is Assumption 6.5 and the main stability and optimality results are demonstrated in Theorem 6.2. A broad range of generation and demand dynamics, including those of higher order, fit within our proposed framework which allows for relaxed stability conditions compared to current literature. Furthermore, it is shown in Proposition 6.1 how the addition of an appropriate observer relaxes the requirement for explicit knowledge of uncontrollable demand within the considered optimality scheme, without compromising stability or optimality. The results are verified with a simulation on the Northeast Power Coordinating Council (NPCC) 140 bus system where convergence to optimality is demonstrated.

Chapter 7 considers secondary frequency regulation with ancillary support from on/off loads. The use of loads that can only take discrete values is in many cases more realistic than a continuous model. In this chapter, power network behaviour is considered at the presence of loads that switch when some prescribed frequency threshold is reached. Hence, loads provide ancillary service at urgencies while their operation is not disrupted at normal operation conditions. First, the case where loads switch on and off at the same frequency threshold is considered, proving convergence of Filippov solutions, as shown in Theorem 7.1. Furthermore, we discuss the possible presence of transient Zeno behaviour phenomena under this scheme, something impractical and hence undesirable. To resolve this issue, loads that switch on and off at different frequency thresholds (i.e. exhibit hysteretic behaviour) are considered. For such schemes, we show that stability guarantees are retained and also that no Zeno behaviour might be exhibited, as demonstrated in Theorem 7.2 and Proposition 7.1. Our analytic results are demonstrated with simulations on the NPCC 140 bus system where it is shown that the presence of switching loads significantly decrease frequency overshoot.

The work of this part is a result of my collaboration with Nima Monshizadeh and Eoin Devane, under the supervision of Dr. Ioannis Lestas. I would like to acknowledge Nima's contribution in both chapters with intuitive and helpful comments and for the development of Lemma 6.4. Furthermore, I would like to acknowledge Eoin's help with insightful discussions in the early development of Chapter 6.

Chapter 6

Stability and optimality of distributed secondary frequency control schemes in power networks

In this chapter, we present a systematic method for designing distributed generation and demand control schemes for secondary frequency regulation in power networks such that stability and an economically optimal power allocation can be guaranteed. A dissipativity condition is imposed on net power supply variables to provide stability guarantees. Furthermore, economic optimality is achieved by explicit decentralised steady state conditions on the generation and controllable demand. We discuss how various classes of dynamics used in recent studies fit within our framework and give examples of higher order generation and controllable demand dynamics that can be included within our analysis. In case of linear dynamics, we discuss how the proposed dissipativity condition can be efficiently verified using an appropriate linear matrix inequality. Moreover, it is shown how the addition of a suitable observer layer can relax the requirement for demand measurements in the employed controller. The efficiency and practicality of the proposed results are demonstrated with a simulation on the Northeast Power Coordinating Council (NPCC) 140-bus system.

6.1 Introduction

It is anticipated that controllable loads will be incorporated within power networks in order to provide benefits such as fast response to changes in power generated from renewable sources and the ability for peak demand reduction. Such changes will greatly increase power network complexity revealing a need for highly distributed schemes that will guarantee its stability when 'plug and play' devices are incorporated. In the recent years, research attention has increasingly focused on such distributed schemes with studies regarding both primary (droop) control as in [42, 111] and secondary control as in [43, 44].

An issue of economic optimality in the power allocation is raised if highly distributed schemes are to be used for frequency control. Recent studies attempted to address this issue by crafting the equilibrium of the system such that it coincides with the optimal solution of a suitable network optimisation problem. To establish optimality of an equilibrium in a distributed fasion, it is evident that a synchronising variable is required. While in the primary control, frequency is used as the synchronising variable (e.g. [60, 61]), in the secondary control a different variable is synchronised by making use of information exchanged between buses [43, 44, 119, 123].

Over the last few years many studies have attempted to address issues regarding stability and optimisation in secondary frequency control. An important feature in many of those is that the dynamics considered follow from a primal/dual algorithm associated with some optimal power allocation problem [43], [81], [82], [124]. This is a powerful approach that reveals the information structure needed to achieve optimality and satisfy the constraints involved. Nevertheless, when higher order generation dynamics need to be considered, these do not necessarily follow as gradient dynamics of a corresponding optimisation problem and therefore alternative approaches need to be employed.

Another trend in the secondary frequency control is the use of distributed averaging proportional integral (DAPI) controllers [59, 85, 58, 125, 86]. DAPI schemes have the advantage that they are simple, since they only measure local frequency and exchange a synchronisation signal in a distributed fashion without requiring load and power flow measurements. On the other hand, it is not easy to accommodate line and power flow constraints, and higher-order generation and controllable demand dynamics in this setting. Moreover the existing results in this context consider only proportional active power sharing and quadratic cost functions.

One of our aims in this chapter is to present a methodology that allows to incorporate general classes of higher order generation and demand control dynamics while ensuring stability and optimality of the equilibrium points. Our analysis borrows ideas from our previous work on primary control, presented in Chapter 4, and adapts those to secondary frequency control, by incorporating the additional communication layer needed in this context. In particular, we consider general classes of aggregate power supply dynamics at each bus and impose two conditions; a dissipativity condition that ensures stability, and a steady-state condition that ensures optimality of the power allocation. An important feature of these conditions is that they are decentralised. Furthermore, in the case of linear supply dynamics, the proposed dissipativity condition can be efficiently verified by means of a linear matrix inequality (LMI). Various examples are also provided to illustrate the significance of our approach and the way it could facilitate a systematic analysis and design. Finally, we discuss how an appropriately designed observer, allows to relax the requirement of an explicit knowledge of the uncontrollable demand, and show that the stability and optimality guarantees remain valid in this case.

The chapter is structured as follows. Section 6.2 provides some basic notation and preliminaries. In Section 6.3 we present the power network model, the classes of generation and controllable demand dynamics and the optimisation problem to be considered. Sections 6.4 and 6.5 include our main assumptions and results. In Section 6.6 we discuss how the results apply to various dynamics for generation and demand, provide intuition regarding our analysis and show how the controller requirements may be relaxed by incorporating an appropriate observer. In Section 6.7, we demonstrate our results through a simulation on the NPCC 140-bus system. Finally, conclusions are drawn in Section 6.8.

6.2 Notation and preliminaries

Most of the notation used within this chapter follows from that introduced in Section 3.1. In addition, for input/output systems $B_j, j = 1, \dots, N$, with respective inputs u_j and outputs y_j, their direct sum, denoted by $\bigoplus_{j=1}^{N} B_j$, represents a system with input $[u_1^T, u_2^T, \dots u_N^T]^T$ and output $[y_1^T, y_2^T, \dots y_N^T]^T$. Furthermore, we write $\mathbf{0}_n$ to denote the $n \times 1$ vector with all elements equal to 0. Finally, for a graph $\mathcal{G} = (N, E)$ we define the directed incidence matrix $\tilde{D}$ to be the $|N| \times |E|$ matrix such that the element $\tilde{D}_{i,j} = -1$ if the edge j leaves node i, $\tilde{D}_{i,j} = 1$ if the edge j enters node i and 0 otherwise.

Within the chapter, we will consider subsystems[1] that will be modelled as dynamical systems with input $u(t) \in \mathbb{R}^m$, state $x(t) \in \mathbb{R}^n$, output $y(t) \in \mathbb{R}^k$ and a

[1]Note that such subsystems will be used to characterise generation and demand dynamics and will be explicitly stated when considered.

state space realisation

$$\begin{aligned} \dot{x} &= f(x, u), \\ y &= g(x, u), \end{aligned} \tag{6.1}$$

where $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ and $g : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous. We assume in (6.1) that given any constant input $u(t) \equiv \bar{u}$, there exists a unique[2] locally asymptotically stable equilibrium point $\bar{x} \in \mathbb{R}^n$, i.e. $f(\bar{x}, \bar{u}) = 0$. The region of attraction[3] of $\bar{x}$ is denoted by X_0. We also define the static input-state characteristic map $k_x : \mathbb{R}^m \to \mathbb{R}^n$ as

$$k_x(\bar{u}) := \bar{x},$$

and the static input-output characteristic map $k_y : \mathbb{R}^m \to \mathbb{R}^k$,

$$k_y(\bar{u}) := g(k_x(\bar{u}), \bar{u}). \tag{6.2}$$

6.3 Problem formulation

6.3.1 Network model

The power network model considered follows from the derivation in Section 2.2.6 and is similar to the model used in the previous chapters. It is described by a connected graph (N, E) where $N = \{1, 2, \ldots, |N|\}$ is the set of buses and $E \subseteq N \times N$ the set of transmission lines connecting the buses. There are two types of buses in the network, buses with inertia and buses without inertia. Since generators have inertia, it is reasonable to assume that only buses with inertia have non-trivial generation dynamics. We define $G = \{1, 2, \ldots, |G|\}$ and $L = \{|G| + 1, \ldots, |N|\}$ as the sets buses with and without inertia respectively such that $|G| + |L| = |N|$. Moreover, the term (i, j) denotes the link connecting buses i and j. The graph (N, E) is assumed to be directed with an arbitrary direction, so that if $(i, j) \in E$ then $(j, i) \notin E$. Additionally, for each $j \in N$, we use $i : i \to j$ and $k : j \to k$ to denote the sets of buses that precede and succeed bus j respectively. It should be noted that the form

[2]The uniqueness assumption on the equilibrium point for a given input could be relaxed to having isolated equilibrium points, but it is used here for simplicity in the presentation.

[3]That is, for the constant input $\zeta_j = \bar{\zeta}_j$, any solution $x(t)$ of (6.4) with initial condition $x(0) \in X_0$ must satisfy $x(t) \to \bar{x}$ as $t \to \infty$.

ω_j	frequency at bus j
η_{ij}	power angle difference between bus i and bus j
p_j^M	mechanical power injection at bus j
d_j^c	controllable load at bus j
d_j^u	uncontrollable frequency dependent load at bus j
p_{ij}	power transfer from bus i to bus j
B_{ij}	line susceptance between bus i and bus j
M_j	generator inertia at bus j
p_j^L	step change in uncontrollable demand at bus j
$x^{M,j}$	internal states of generation dynamics at bus j
$x^{c,j}$	internal states of controllable load dynamics at bus j
$x^{u,j}$	internal states of uncontrollable frequency dependent load dynamics at bus j
s_j	net power supply at bus j
p_j^c	power command at bus j
ψ_{ij}	integral of power command difference between bus i and bus j

Table 6.1. Notation used in the system model (6.3)–(6.6). Note that variables ω_j, p_j^M, d_j^c, d_j^u, p_j^L, s_j, p_j^c and ψ_{ij} denote deviations from corresponding nominal values. The internal states are the states in the state space representation of the differential equations representing the dynamics (details can be found in sections 6.2 and 6.3).

of the dynamics in (6.3)–(6.4) below is not affected by changes in graph ordering, and our results are independent of the choice of direction. We make the following assumptions for the network:

1) Bus voltage magnitudes are $|V_j| = 1$ p.u. for all $j \in N$.
2) Lines $(i,j) \in E$ are lossless and characterised by their susceptances $B_{ij} = B_{ji} > 0$.
3) Reactive power flows do not affect bus voltage phase angles and frequencies.

Such assumptions are generally valid at medium to high voltages or when tight voltage control is present, and are often used in secondary frequency control studies [113].

Swing equations can then be used to describe the rate of change of frequency at generation buses. Power must also be conserved at each of the load buses. This motivates the following system dynamics (e.g. [113]),

$$\dot{\eta}_{ij} = \omega_i - \omega_j,\ (i,j) \in E, \tag{6.3a}$$

$$M_j \dot{\omega}_j = -p_j^L + p_j^M - (d_j^c + d_j^u) - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij},\ j \in G, \tag{6.3b}$$

$$0 = -p_j^L - (d_j^c + d_j^u) - \sum_{k:j \to k} p_{jk} + \sum_{i:i \to j} p_{ij},\ j \in L, \tag{6.3c}$$

$$p_{ij} = B_{ij} \sin \eta_{ij} - p_{ij}^{nom},\ (i,j) \in E. \tag{6.3d}$$

In system (6.3), the time-dependent variables ω_j, d_j^c and p_j^M represent, respectively, deviations from a nominal value[4] for the frequency and controllable load at bus j and the mechanical power injection to the generation bus j. The quantity d_j^u represents the uncontrollable frequency-dependent load and generation damping present at bus j. The time-dependent variables η_{ij} and p_{ij} represent, respectively, the power angle difference[5] and the deviation of the power transferred from bus i to bus j from the nominal value, p_{ij}^{nom}. The constant $M_j > 0$ denotes the generator inertia. The response of the system (6.3) will be studied, when a step change $p_j^L, j \in N$ occurs in the uncontrollable demand.

In order to investigate broad classes of generation and demand dynamics and control policies, we let the scalar variables p_j^M, d_j^c, and d_j^u be generated by dynamical systems of form (6.1), namely

$$\begin{aligned} \dot{x}^{M,j} &= f^{M,j}(x^{M,j}, \zeta_j), \\ p_j^M &= g^{M,j}(x^{M,j}, \zeta_j), \end{aligned} \qquad j \in G, \tag{6.4a}$$

$$\begin{aligned} \dot{x}^{c,j} &= f^{c,j}(x^{c,j}, \zeta_j), \\ d_j^c &= g^{c,j}(x^{c,j}, \zeta_j), \end{aligned} \qquad j \in N, \tag{6.4b}$$

$$\begin{aligned} \dot{x}^{u,j} &= f^{u,j}(x^{u,j}, -\omega_j), \\ -d_j^u &= g^{u,j}(x^{u,j}, -\omega_j), \end{aligned} \qquad j \in N \tag{6.4c}$$

where the input ζ_j is defined as $\zeta_j = [-\omega_j\ p_j^c]^T$ with p_j^c representing the deviations of a power command signal from its nominal value. Notice that in the case of uncontrollable demand, the input is given in terms of the local frequency deviation ω_j only, and is decoupled from the power command signal as expected.

For notational convenience, we collect the variables in (6.4) into the vectors $x^M = [x^{M,j}]_{j \in G}$, $x^c = [x^{c,j}]_{j \in N}$, and $x^u = [x^{u,j}]_{j \in N}$. These quantities represent the

[4]A nominal value of a variable is defined as its value at an equilibrium of (6.3) with frequency at its nominal value of 50Hz (or 60Hz).

[5]The quantities η_{ij} represent the phase differences between buses i and j, given by $\theta_i - \theta_j$, i.e. $\eta_{ij} = \theta_i - \theta_j$. The angles themselves must also satisfy $\dot{\theta}_j = \omega_j$ at all $j \in N$. This equation is omitted in (6.3) since the power transfers are functions of the phase differences only.

internal states of the dynamical systems used to update the outputs p_j^M, d_j^c, and d_j^u.

In terms of the outputs from (6.4), it will be useful to consider the net supply variables s, defined as

$$s_j = p_j^M - d_j^c,\ j \in G, \qquad s_j = -d_j^c,\ j \in L. \tag{6.5}$$

The variables defined in (6.5) evolve according to the dynamics described in (6.4a) - (6.4b). Therefore, s_j are outputs from these combined controlled dynamical systems with inputs ζ_j.

6.3.2 Power command dynamics

We consider a communication network described by a connected graph $(N, \tilde{E})$, where $\tilde{E}$ represents the set of communication lines among the buses, i.e., $(i,j) \in \tilde{E}$ if buses i and j communicate. Note that $\tilde{E}$ can be different from the set of flow lines E. We will study the behaviour of the system (6.3)–(6.4) under the following dynamics for the power command signal p_j^c which has been used in literature (e.g. [43, 81]),

$$\gamma_{ij}\dot{\psi}_{ij} = p_i^c - p_j^c,\ (i,j) \in \tilde{E} \tag{6.6a}$$

$$\gamma_j \dot{p}_j^c = -(s_j - p_j^L) - \sum_{k:j\to k} \psi_{jk} + \sum_{i:i\to j} \psi_{ij},\ j \in N \tag{6.6b}$$

where γ_j and γ_{ij} are positive constants, and the variable ψ_{ij} represents the difference in the integrals between the power commands of communicating buses i and j. It should be noted that p_i^c and p_j^c are variables shared between communicating buses i and j.

Although the dynamics in (6.6) do not directly integrate frequency, we will see later that under a weak condition on the steady state behaviour of d^u, they guarantee convergence to the nominal frequency for a broad class of supply dynamics. The dynamics in (6.6), often referred as 'virtual swing equations', are frequently used in the literature[6] as they achieve both the synchronisation of the communicated variable p^c, something that can be exploited to guarantee optimality of the equilibrium point reached, and also the convergence of frequency to its nominal value.

[6]We use for simplicity a single communicating variable. It should be noted that more advanced communication structures (e.g. [43]) can allow additional constraints to be satisfied in the optimisation problem posed.

6.3.3 Optimal generation and load control

We aim to study how generation and controllable demand should be adjusted in order to meet the step change in frequency independent demand and at the same time minimise the resulting cost from the deviation in the power generated and the disutility of loads. Below, we introduce an optimisation problem, which we call the optimal generation and load control problem (OGLC), that can be used to achieve this goal.

It is supposed that a cost $C_j(p_j^M)$ is induced when generation output at bus j is changed by p_j^M from its nominal value. Similarly, a cost of $C_{dj}(d_j^c)$ is incurred for a change of d_j^c in controllable demand. The total cost within OGLC is the sum of the above costs. The problem is to find the vectors p^M and d^c that minimize this total cost and simultaneously achieve power balance, while satisfying physical saturation constraints. More precisely, the following optimisation problem is considered

$$
\begin{aligned}
&\underline{\text{OGLC:}}\\
&\quad \min_{p^M, d^c} \sum_{j \in G} C_j(p_j^M) + \sum_{j \in N} C_{dj}(d_j^c),\\
&\quad \text{subject to } \sum_{j \in G} p_j^M = \sum_{j \in N} (d_j^c + p_j^L),\\
&\quad p_j^{M,min} \leq p_j^M \leq p_j^{M,max}, \ \forall j \in G,\\
&\quad d_j^{c,min} \leq d_j^c \leq d_j^{c,max}, \ \forall j \in N,
\end{aligned}
\tag{6.7}
$$

where $p_j^{M,min}, p_j^{M,max}, d_j^{c,min}$, and $d_j^{c,max}$ are bounds for the minimum and maximum values for generation and controllable demand deviations, respectively, at bus j. The equality constraint in (6.7) requires all the additional frequency-independent loads to be matched by the total deviation in generation and controllable demand. This ensures that when system (6.3) is at equilibrium and a mild condition described in Assumption 6.3 below holds, the frequency will be at its nominal value.

Within the chapter we aim to specify properties on the control dynamics of p^M and d^c, described in (6.4a)–(6.4b), that ensure that those quantities converge to values at which optimality can be guaranteed for (6.7).

The assumption below allows the use of the KKT conditions to prove the optimality result in Theorem 6.1 in Section 6.5.

Assumption 6.1 *The cost functions C_j and C_{dj} are continuously differentiable and strictly convex.*

6.3.4 Equilibrium analysis

We now describe what is meant by an equilibrium of the interconnected system (6.3)–(6.6).

Definition 6.1 *The point* $\beta^* = (\eta^*, \psi^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*}, p^{c,*})$ *defines an equilibrium of the system* (6.3)–(6.6) *if all time derivatives of* (6.3)–(6.6) *are equal to zero at this point.*

It should be noted that the static input-output maps $k_{p_j^M}$, $k_{d_j^c}$, and $k_{d_j^u}$, as defined in (6.2), completely characterise the equilibrium behaviour of (6.4). In our analysis, we shall consider conditions on these characteristic maps relating input $\zeta_j = [-\omega_j \; p_j^c]^T$ and generation/demand such that their equilibrium values are optimal for (6.7), thus making sure that frequency will be at its nominal value at steady state.

Throughout the chapter, it is assumed that there exists some equilibrium of (6.3)–(6.6) as defined in Definition 6.1. Any such equilibrium is denoted by $\beta^* = (\eta^*, \psi^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*}, p^{c,*})$. Furthermore, we use $(p^*, p^{M,*}, d^{c,*}, d^{u,*}, \zeta^*, s^*)$ to represent the equilibrium values of respective quantities in (6.3)–(6.6).

The power angle differences at the considered equilibrium are assumed to satisfy the following security constraint.

Assumption 6.2 $|\eta_{ij}^*| < \frac{\pi}{2}$ *for all* $(i,j) \in E$.

Moreover, the following assumption is related with the steady state values of variable d^u, describing uncontrollable demand and generation damping. It is a mild condition associated with having negative feedback from d^u to frequency.

Assumption 6.3 *For each* $j \in N$, *the functions* $k_{d_j^u}$ *relating the steady state values of frequency and uncontrollable loads satisfy* $\bar{u}_j k_{d_j^u}(\bar{u}_j) > 0$ *for all* $\bar{u}_j \in \mathbb{R} - \{0\}$.

Although not required for stability, Assumption 6.3 guarantees that the frequency will be equal to its nominal value at equilibrium, i.e. $\omega^* = \mathbf{0}_{|N|}$, as stated in the following lemma.

Lemma 6.1 *Let Assumption 6.3 hold. Then, any equilibrium point* β^* *given by Definition 6.1 satisfies* $\omega^* = \mathbf{0}_{|N|}$.

The stability and optimality properties of such equilibria will be studied in the following sections.

6.3.5 Additional conditions

Due to the fact that the frequency at the load buses is related with the system states by means of algebraic equations, additional conditions are needed for the system (6.3)–(6.4) to be well-defined. We use below the vector notation $\omega^G = [\omega_j]_{j\in G}$ and $\omega^L = [\omega_j]_{j\in L}$.

Assumption 6.4 *There exists an open neighbourhood T of $(\eta^*, \omega^{G,*}, x^{c,*}, x^{u,*}, p^{c,*})$ and a locally Lipschitz map f^L such that when $(\eta, \omega^G, x^c, x^u, p^c) \in T$, it holds that $\omega^L = f^L(\eta, \omega^G, x^c, x^u, p^c)$.*

Remark 6.1 *Assumption 6.4 is a technical assumption that is required in order for the system* (6.3)–(6.4) *to have a locally well-defined state space realisation. It can often be easily verified by means of the implicit function theorem [126]. Without Assumption 6.4, stability could be studied through more technical approaches such as the singular perturbation analysis discussed in [127, Section 6.4].*

6.4 Dissipativity conditions on generation and demand dynamics

Before we state our main results in Section 6.5, it would be useful to provide a dissipativity definition, based on [128], for systems of the form (6.1). This notion will be used to formulate appropriate decentralised conditions on the uncontrollable demand and power supply dynamics (6.4c), (6.5).

Definition 6.2 *The system* (6.1) *is said to be locally dissipative about the constant input values $\bar{u}$ and corresponding equilibrium state values $\bar{x}$, with supply rate function $W : \mathbb{R}^{n+k} \to \mathbb{R}$, if there exist open neighbourhoods U of $\bar{u}$ and X of $\bar{x}$, and a continuously differentiable, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ (called the storage function), with a strict local minimum at $x = \bar{x}$, such that for all $u \in U$ and all $x \in X$,*

$$\dot{V}(x, u) \leq W(u, g(x, u)). \tag{6.8}$$

We now assume that the systems with input $\zeta_j = [-\omega_j \quad p_j^c]^T$ and output the power supply variables and uncontrollable loads satisfy the following local dissipativity condition.

Assumption 6.5 *The systems with inputs $\zeta_j = [-\omega_j \; p_j^c]^T$ and outputs $y_j = [s_j \; -d_j^u]^T$ described in* (6.5) *and* (6.4c) *satisfy a dissipativity condition about constant* *input values ζ_j^* and corresponding equilibrium state values $(x^{M,j,*}, x^{c,j,*}, x^{u,j,*})$ in the sense of Definition 6.2, with supply rate functions*

$$W_j(\zeta_j, y_j) = [(s_j - s_j^*)\ (-d_j^u - (-d_j^{u,*}))] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (\zeta_j - \zeta_j^*) - \phi_j(\zeta_j - \zeta_j^*),\ j \in N, \tag{6.9}$$

where $\phi_j : \mathbb{R}^2 \to \mathbb{R}$. Furthermore, one of the following two properties holds,

(a) The function ϕ_j is positive definite.

(b) The function ϕ_j is positive semidefinite and positive definite with respect to ω_j. Also when ω_j, s_j are constant for all times then p_j^c cannot be a nontrivial sinusoid[7]*.*

We shall refer to Assumption 6.5 when condition (a) holds for ϕ_j as Assumption 6.5(a) (respectively Assumption 6.5(b) when (b) holds).

Remark 6.2 *Assumption 6.5 is a decentralised condition that allows to incorporate a broad class of generation and load dynamics, including various examples that have been used in the literature (these will be discussed in Section 6.6). Furthermore, for linear systems Assumption 6.5 can be formulated as the feasibility problem of a corresponding LMI (linear matrix inequality) [129], and it can therefore be verified by means of computationally efficient methods.*

Remark 6.3 *Condition (b) in Assumption 6.5 is a relaxation of condition (a) whereby ϕ is not required to be positive definite. This permits the inclusion of a broader class of dynamics from p_j^c to s_j as it will be discussed in Section 6.6. However, it requires that the power command p^c cannot be a sinusoid if both s_j and ω_j are constant. This additional condition is necessary as the dynamics in* (6.6) *allow p_j^c to be a sinusoid when s_j is constant. For linear systems, this condition is implied by the rather mild assumption that no imaginary axis zeros are present in the transfer function from p_j^c to s_j. Furthermore, although it is difficult to test Assumption 6.5(b) on general non-linear systems, it can be seen to apply to highly relevant non-linear dynamics, such as those described in* (6.12) *within Section 6.6.*

[7]By nontrivial sinusoid, we mean functions of the form $\sum_j A_j \sin(\omega_j t + \phi_j)$ that are not equal to a constant.

Remark 6.4 *Further intuition on the dissipativity condition in Assumption 6.5 will be provided in Section 6.6.1. In particular, it will be discussed that when $\phi_j = 0$ that this is a decentralised condition that is necessary and sufficient for the passivity of an appropriately defined multivariable system quantifying aggregate dynamics at each bus.*

6.5 Main results

In this section we state our main results, proven within Appendix A. Our first result provides conditions for the equilibrium points to be solutions[8] to the OGLC problem (6.7).

Theorem 6.1 *Suppose that Assumption 6.1 is satisfied and the control dynamics in* (6.4a) *and* (6.4b) *are chosen such that*

$$\begin{aligned} k_{p_j^M}([0\ p_j^c]^T) &= [(C_j')^{-1}(p_j^c)]_{p_j^{M,min}}^{p_j^{M,max}} \\ k_{d_j^c}([0\ p_j^c]^T) &= [(C_{dj}')^{-1}(p_j^c)]_{d_j^{c,min}}^{d_j^{c,max}} \end{aligned} \tag{6.10}$$

holds. Then, the equilibrium values $p^{M,*}$ and $d^{c,*}$ are optimal solutions to the OGLC problem (6.7)*.*

Our second result shows that the set of equilibria for the system described by (6.3)–(6.6) for which Assumptions 6.1 - 6.5 are satisfied is asymptotically attracting, the equilibria are global minima of the OGLC problem (6.7) and, as shown in Lemma 6.1, satisfy $\omega^* = \mathbf{0}_{|N|}$.

Theorem 6.2 *Consider equilibria of* (6.3)–(6.6) *with respect to which Assumptions 6.1–6.5 are all satisfied. If the control dynamics in* (6.4a) *and* (6.4b) *are chosen such that* (6.10) *holds, then there exists an open neighbourhood of initial conditions about any such equilibrium such that the solutions of* (6.3)–(6.6) *are guaranteed to converge to a set of equilibria that solve the OGLC problem* (6.7) *with $\omega^* = \mathbf{0}_{|N|}$.*

6.6 Discussion

In this section we discuss examples that fit within the framework presented in this

[8]Note that an equilibrium point is a solution to the OGLC problem when at that point the variables that appear in (6.7) are solutions to the problem.

chapter, and also describe how the dissipativity condition of Assumption 6.5 can be verified for linear systems via a linear matrix inequality.

We start by giving various examples of power supply dynamics that have been used in the literature that satisfy our proposed dissipativity condition in Assumption 6.5. Consider the load models used in [43], [123], and [81], where the power supply is a static function of ω_j and p^c_j,

$$s_j = (C'_j)^{-1}(p^c_j - \omega_j),\ j \in N, \tag{6.11}$$

where C_j is some convex cost function, and generation damping/uncontrollable demand is given by $d^u_j = \lambda_j \omega_j, \lambda_j > 0$. It is easy to show that Assumption 6.5(a) holds for these widely used schemes.

Furthermore, Assumption 6.5(b) is satisfied when first order generation dynamics are used such as

$$\dot{s}_j = -\mu_j(C'_j(s_j) - (p^c_j - \omega_j)) \tag{6.12}$$

with $d^u_j = \lambda_j \omega_j$ and $\lambda_j, \mu_j > 0$. Such first order models have often been used in the literature as in [124].

A significant aspect of the framework presented in this chapter is that it also allows higher order dynamics for the power supply to be incorporated. As an example, we consider the following second-order model,

$$\begin{aligned} \dot{\alpha}_j &= -\frac{1}{\tau_{a,j}}(\alpha_j - K_j(p^c_j - \omega_j)), \\ \dot{z}_j &= -\frac{1}{\tau_{b,j}}(z_j - \alpha_j), \\ s_j - d^u_j &= z_j - \lambda_j \omega_j + \lambda^{PC}_j p^c_j, \end{aligned} \tag{6.13}$$

where α_j, z_j are states and $\tau_{a,j}, \tau_{b,j} > 0$ time constants associated with the turbine-governor dynamics, $\lambda_j > 0$ is a damping coefficient[9], constant $K_j > 0$ determines the strength of the feedback gain, and the term $\lambda^{PC}_j p^c_j$ represents static dependence on power command due to either generation or controllable loads[10]. It can be shown

[9]Note that the term $\lambda_j \omega_j$ can be incorporated in s_j or d^u_j.

[10]It should be noted that the term d^u_j can also include controllable demand and generation that depend on frequency only (i.e. not on power command). Therefore, d^u_j can be perceived to contain all frequency dependent terms that return to their nominal value at steady state and therefore do not contribute to secondary frequency control.

that Assumption 6.5 is satisfied for all $\tau_{a,j}, \tau_{b,j} > 0$ when[11] $K_j < 8\lambda_j^{PC}$ and $\lambda_j^{PC} \leq \lambda_j$.

Another feature of Assumption 6.5 is that it can be efficiently verified for a general linear system by means of an LMI, i.e. a computationally efficient convex problem. In particular, it can be shown [129] that if the system in Assumption 6.5 is linear with a minimal state space realisation

$$\begin{aligned} \dot{x} &= Ax + B\tilde{u}, \\ \tilde{y} &= Cx + D\tilde{u}, \end{aligned} \tag{6.14}$$

where $\tilde{u} = \zeta - \zeta^*$ and $\tilde{y} = y - y^*$, and ϕ_j is chosen as a quadratic function $\phi_j = \epsilon_1(\omega_j - \omega_j^*)^2 + \epsilon_2(p_j^c - p_j^{c,*})^2$ with[12] $\epsilon_1, \epsilon_2 > 0$ then the dissipativity condition in Assumption 6.5 is satisfied if and only if there exists $P = P^T \geq 0$ such that

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} - \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T Q \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \leq 0, \tag{6.15}$$

where the matrix Q is given by

$$Q = \begin{bmatrix} 0 & M \\ M & K \end{bmatrix}, \; M = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \; K = \begin{bmatrix} -\epsilon_1 & 0 \\ 0 & -\epsilon_2 \end{bmatrix}.$$

This approach could also be exploited to form various convex optimisation problems that could facilitate design. For example, one could obtain the minimum damping such that Assumption 6.5 is satisfied at a bus.

To further demonstrate the applicability of our approach we consider a fifth order model for turbine governor dynamics provided by the Power System Toolbox [115]. The dynamics are described by the following transfer function relating the mechanical power supply[13] $\hat{s}_j$ with the negative frequency deviation $-\hat{\omega}_j$,

$$G_j(s) = K_j \frac{1}{(1 + sT_{s,j})} \frac{(1 + sT_{3,j})}{(1 + sT_{c,j})} \frac{(1 + sT_{4,j})}{(1 + sT_{5,j})},$$

where K_j and $T_{s,j}, T_{3,j}, T_{c,j}, T_{4,j}, T_{5,j}$ are the droop coefficient and time-constants respectively. Realistic values for these models are provided by the toolbox for the

[11] A second order model was studied for a related problem in [82], with the stability condition requiring, roughly speaking, that the gain of the system is less than the damping provided by the loads. The LMI approach described in this section allows such conditions to be relaxed.

[12] We could also have $\epsilon_2 = 0$ if (6.14) has no zeros on the imaginary axis, as stated in condition (b) for ϕ in Assumption 6.5, and Remark 6.3.

[13] Note that $\hat{s}_j$ denotes the Laplace transform of s_j.

NPCC network, with turbine governor dynamics implemented on 22 buses. The corresponding buses also have appropriate frequency damping λ_j. We examined the effect of incorporating a power command input signal in the above dynamics by considering the supply dynamics

$$\hat{s}_j - \hat{d}^u_j = (G_j + \lambda_j)(-\hat{\omega}_j) + (G_j + \lambda^{PC}_j)(\hat{p}^c_j), j \in N$$

where $\lambda^{PC}_j > 0, j \in N$ is a coefficient representing the static dependence on power command. For appropriate values of λ^{PC}_j, the condition in Assumption 6.5 was satisfied for 20 out of the 22 buses, while for the remaining 2 buses the damping coefficients λ_j needed to be increased by 37% and 28% respectively. Furthermore, filtering the power command signal with appropriate compensators, allowed a significant decrease in the required value for λ^{PC}_j. Power command and frequency compensation may also be used with alternative objectives, such as to improve the stability margins and system performance.

The fact that our condition is satisfied at all but two buses[14], demonstrates that it is not conservative in existing implementations. Note also that a main feature of this condition is the fact that it is decentralised, involving only local bus dynamics, which can be important in practical implementations.

6.6.1 System representation

To further illustrate the described system dynamics, we have provided a schematic representation on Fig. 6.1, representing equations (6.3)–(6.6). The figure clarifies the interconnection among the various control dynamics in the power network. Pure integrators are denoted by $\frac{1}{s}$, and $D = \text{blockdiag}\,(D_p, D_c)$ is a block diagonal matrix with its blocks being the incidence matrices of the power and communication networks D_p and D_c respectively. Note that the vectors p^{net} and ψ^{net} represent the aggregate power transfer and summation of ψ variables at each bus and have respective j^{th} components $\sum_{i:i\to j} p_{ij} - \sum_{k:j\to k} p_{jk}$ and $\sum_{i:i\to j} \psi_{ij} - \sum_{k:j\to k} \psi_{jk}$.

It is useful and intuitive to note that the system (6.3) - (6.6) considered within the chapter can be represented by a negative feedback interconnection of systems I and $B = \bigoplus_{j=1}^{|N|} B_j$, containing all interconnection and bus dynamics respectively. More precisely, I and B have respective inputs u^I and u^B, and respective outputs

[14] Note that this is satisfied at all buses with appropriate increase in damping.

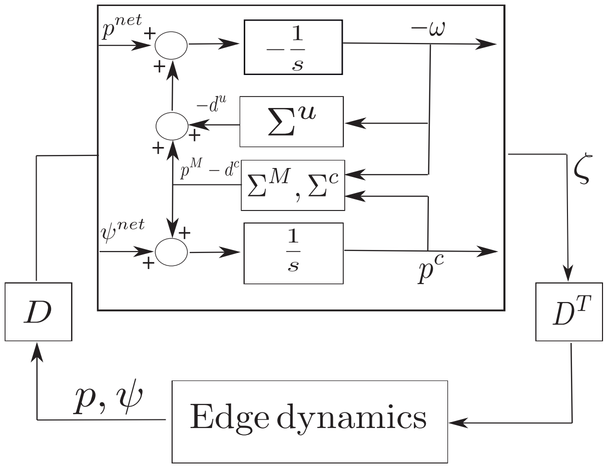

Figure 6.1. Schematic overview of the system described by (6.3)–(6.6).

u^B and $-u^I$, defined as

$$u^I = \begin{bmatrix} \omega_1 \\ -p_1^c \\ \dots \\ \omega_{|N|} \\ -p_{|N|}^c \end{bmatrix}, \; u^B = \begin{bmatrix} \sum_{k:1\to k} p_{1k} - \sum_{i:i\to 1} p_{i1} \\ \sum_{i:i\to 1} \psi_{i1} - \sum_{k:1\to k} \psi_{1k} \\ \dots \\ \sum_{k:|N|\to k} p_{|N|k} - \sum_{i:i\to |N|} p_{i|N|} \\ \sum_{i:i\to |N|} \psi_{i|N|} - \sum_{k:|N|\to k} \psi_{|N|k} \end{bmatrix}.$$

The subsystems B_j, representing the dynamics at bus j, have inputs u_j and outputs y_j described by,

$$u_j = \begin{bmatrix} \sum_{k:j\to k} p_{jk} - \sum_{i:i\to j} p_{ij} \\ \sum_{i:i\to j} \psi_{ij} - \sum_{k:j\to k} \psi_{jk} \end{bmatrix}, y_j = \begin{bmatrix} \omega_j \\ -p_j^c \end{bmatrix}. \tag{6.16}$$

Note that system B is depicted by the upper block on Fig. 6.1 and system I by the rest three blocks within the figure.

It can easily be shown that System I is locally passive[15]. The following theorem shows that Assumption 6.5 with $\phi = 0$ is sufficient for the passivity of each individual subsystem B_j.

Theorem 6.3 *Let Assumption 6.5 hold with $\phi_j = 0$ about an equilibrium. Then the corresponding subsystem B_j with inputs and outputs given by* (6.16) *is passive about that equilibrium.*

[15]By a locally passive system we refer to a system satisfying the dissipativity condition in Definition 6.2 with the supply rate being $W(u, y) = (u - u^*)^T (y - y^*)$.

Remark 6.5 *The significance of the interpretation discussed in this section is that the passivity property of system I, in conjunction with the fact that $B = \bigoplus_{j=1}^{|N|} B_j$, implies that stability of the network is guaranteed if the subsystems B_j are passive (with appropriate strictness as quantified within the chapter). In particular, stability is guaranteed in a decentralised way without requiring information about the rest of the network at each individual bus, which is advantageous in highly distributed schemes where a "plug and play" capability is needed. It should be noted that the subsystems B_j are multivariable systems quantifying the aggregate bus dynamics associated with both power generation and the communicated signal p^c.*

Remark 6.6 *It is shown in Appendix B that Assumption 6.5 is also necessary for systems B_j to be passive, for general affine nonlinear dynamics. Hence, Assumption 6.5 introduces no additional conservatism in this property for a large class of nonlinear systems.*

6.6.2 Observing uncontrollable frequency independent demand

The power command dynamics in (6.6) involve the uncontrollable frequency independent demand p^L. We discuss in this section that the inclusion of appropriate observer dynamics for p^L allows convergence to optimality to be achieved when p^L is not directly known.

A way to obtain p^L, could be by re-arranging equations (6.3b)–(6.3c). This approach would require knowledge of power supply and power transfers in load buses, which is realistic. However, knowledge of the frequency derivative would also be required for its estimation at generation buses, which might be difficult to obtain in noisy environments.

We therefore consider instead observer dynamics[16] for p^L_j that are incorporated within the power command dynamics. In particular the following dynamics are considered

$$\gamma_{ij}\dot{\psi}_{ij} = p^c_i - p^c_j, \ (i,j) \in \tilde{E}, \tag{6.17a}$$

$$\gamma_j \dot{p}^c_j = -(s_j - \chi_j) - \sum_{k:j\to k} \psi_{jk} + \sum_{i:i\to j} \psi_{ij}, \ j \in N, \tag{6.17b}$$

[16] See also the use of observer dynamics in [130] as a means of counteracting agent dishonesty.

$$\tau_{\chi,j}\dot{\chi}_j = b_j - \omega_j - p_j^c - \chi_j,\ j \in G, \tag{6.17c}$$

$$M_j\dot{b}_j = -\chi_j + s_j - d_j^u - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij},\ j \in G, \tag{6.17d}$$

$$0 = -\chi_j + s_j - d_j^u - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij},\ j \in L, \tag{6.17e}$$

where $\tau_{\chi,j}$ are positive time constants and b_j and χ_j are auxiliary variables associated with the observer.

The equilibria of the system (6.3) – (6.5), (6.17) are defined in a similar way to Definition 6.1 and it is assumed that at least one such equilibrium exists. Note that the existence of an equilibrium of (6.3) - (6.6) implies the existence of an equilibrium of (6.3)–(6.5), (6.17).

We now provide a result analogous to Lemma 6.1 in the case where the observer dynamics are included. Lemma 6.2, as well as Proposition 6.1 below are proven in Appendix A.

Lemma 6.2 *Let Assumption 6.3 hold. Then, any equilibrium point* $(\eta^*, \psi^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*}, p^{c,*}, b^*, \chi^*)$ *of the system* (6.3) – (6.5), (6.17) *satisfies* $\omega^* = \mathbf{0}_{|N|}$.

Remark 6.7 *The dynamics in* (6.17) *eliminate the requirement to explicitly know* p^L *within the power command dynamics by adding an observer that mimics the swing equation, described by* (6.17c)–(6.17e). *The dynamics in* (6.17d)–(6.17e) *ensure that the variable* χ_j *is equal at steady state to the value* $\chi_j^* = s_j^* - d_j^{u,*} - \sum_{k:j\to k} p_{jk}^* + \sum_{i:i\to j} p_{ij}^* = p_j^L$ *for* $j \in N$*, with the second part of the equality coming from* (6.3b)–(6.3c) *at equilibrium. As shown in Lemma 6.2, such equilibrium guarantees that the steady state value of the frequency will be equal to the nominal one.*

The following proposition shows that the set of equilibria for the system described by (6.3) – (6.5), (6.17) for which Assumptions 6.1 - 6.5 are satisfied is asymptotically attracting and that these equilibria are also solutions to the OGLC problem (6.7).

Proposition 6.1 *Consider equilibria of* (6.3) – (6.5), (6.17) *with respect to which Assumptions 6.1–6.5 are all satisfied. If the control dynamics in* (6.4a) *and* (6.4b) *are chosen such that* (6.10) *is satisfied then there exists an open neighbourhood of initial conditions about any such equilibrium such that the solutions of* (6.3) – (6.5), (6.17) *are guaranteed to converge to a global minimum of the OGLC problem* (6.7) *with* $\omega^* = \mathbf{0}_{|N|}$.

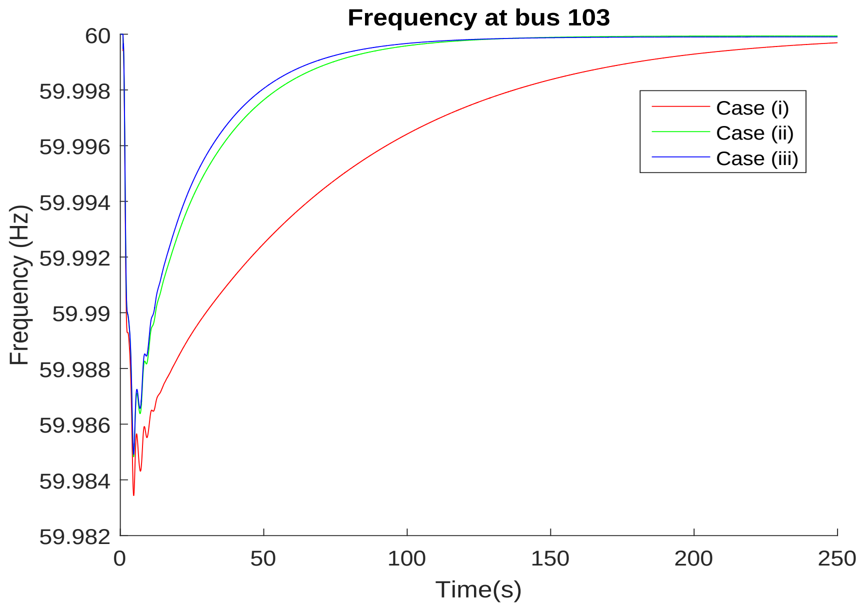

Figure 6.2. Frequency at bus 103 with: i) 10 generators, ii) 10 generators and 20 controllable loads, iii) 15 generators and 20 controllable loads, contributing to secondary frequency control.

Remark 6.8 *Note that in some cases there could be uncertainty in the knowledge of the d^u dynamics. This does not affect the optimality of the equilibrium points since at equilibrium we have $d^u = \mathbf{0}_{|N|}$. Numerical simulations with realistic data have demonstrated that network stability is also robust to variations in the d^u model used in* (6.17d)–(6.17e).

6.7 Simulation on the NPCC 140-bus system

In this section we use the Northeast Power Coordinating Council (NPCC) 140-bus interconnection system, simulated using the Power System Toolbox [115], in order to illustrate our results. This model is more detailed and realistic than our analytical one, including line resistances, a DC12 exciter model, a subtransient reactance generator model, and higher order turbine governor models[17].

The test system consists of 93 load buses serving different types of loads including constant active and reactive loads and 47 generation buses. The overall system has a total real power of 28.55GW. For our simulation, we added three loads on units 2, 9, and 17, each having a step increase of magnitude 1 p.u. (base 100MVA) at $t = 1$ second.

[17]The details of the simulation models can be found in the Power System Toolbox data file datanp48.

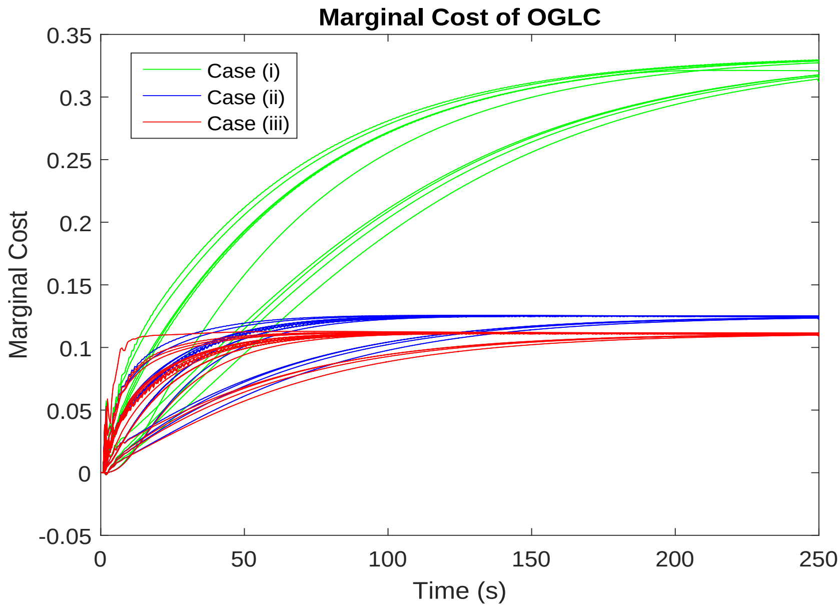

Figure 6.3. Marginal costs for controllable loads and generators with non-equal cost coefficients for the three test cases.

Controllable demand was considered within the simulations, with loads controlled every 10ms. The disutility function for the deviation d_j^c in controllable loads in each bus was $C_{dj}(d_j^c) = \frac{1}{2}\alpha_j(d_j^c)^2$. The selected values for cost coefficients were $\alpha_j = 1$ for load buses $1-5$ and $11-15$ and $\alpha_j = 2$ for the rest. Similarly, the cost functions for deviations p_j^M in generation were $C_j(p_j^M) = \frac{1}{2}\kappa_j(p_j^M)^2$, where κ_j were selected as the inverse of the generators droop coefficients, as suggested in (6.10).

Consider the static and first order dynamic schemes given by $d_j^c = (C'_{dj})^{-1}(\omega_j - p_j^c)$ and $\dot{d}_j^c = -d_j^c + (C'_{dj})^{-1}(\omega_j - p_j^c), j \in N$, where p_j^c has dynamics as described in (6.6). We refer to the resulting dynamics as Static and Dynamic OGLC respectively since in both cases, steady state conditions that solve the OGLC problem were used. As discussed in Section 6.6, in the presence of arbitrarily small frequency damping, both schemes satisfy Assumption 6.5 and are thus included in our framework.

The system was tested on three different cases. In case (i) 10 generators were employed to perform secondary frequency control by having frequency and power command as inputs. In case (ii) controllable loads were included on 20 load buses in addition to the 10 generators. Controllable load dynamics in 10 buses were described by Static OGLC and in the rest by Dynamic OGLC. Finally, in case (iii), all controllable loads of case (ii) and 15 generators where used for secondary frequency control. Note that the 15 generators used for secondary frequency control had third, fourth and fifth order turbine governor dynamics.

The frequency at bus 103 for the three tested cases is shown in Fig. 6.2. From this figure, we observe that in all cases the frequency returns to its nominal value. However, the presence of controllable loads makes the frequency return much faster and with a smaller overshoot.

Furthermore, from Fig. 6.3, it is observed that the marginal costs at all controlled loads and generators that contribute to secondary frequency control, converge to the same value. This illustrates the optimality in the power allocation among generators and loads, since equality in the marginal cost is necessary to solve (6.7) when the power generated does not saturate to its maximum/minimum value.

6.8 Conclusion

We have considered the problem of designing distributed schemes for secondary frequency control such that stability and optimality of the power allocation can be guaranteed. In particular, we have considered general classes of generation and demand control dynamics and have shown that a dissipativity condition in conjunction with appropriate decentralised conditions on their steady state behaviour can provide such stability and optimality guarantees. We have also discussed that for linear systems the dissipativity condition can be easily verified by solving a corresponding LMI and shown that the requirement to have knowledge of demand may be relaxed by incorporating an appropriate observer. Our results have been illustrated with simulations on the NPCC 140-bus system.

Appendix A

In this appendix we prove our main results, Theorems 6.1 - 6.2, and also Lemmas 6.1-6.2, Theorem 6.3 and Proposition 6.1.

Throughout the proofs we will make use of the following equilibrium equations for the dynamics in (6.3)–(6.4),

$$0 = \omega_i^* - \omega_j^*,\ (i,j) \in E, \tag{6.18a}$$

$$0 = -p_j^L + p_j^{M,*} - (d_j^{c,*} + d_j^{u,*}) - \sum_{k:j\to k} p_{jk}^* + \sum_{i:i\to j} p_{ij}^*,\ j \in G, \tag{6.18b}$$

$$0 = -p_j^L - (d_j^{c,*} + d_j^{u,*}) - \sum_{k:j\to k} p_{jk}^* + \sum_{i:i\to j} p_{ij}^*,\ j \in L, \tag{6.18c}$$

$$p_j^{M,*} = k_{p_j^M}(\zeta_j^*),\ j \in G, \tag{6.18d}$$

$$d_j^{c,*} = k_{d_j^c}(\zeta_j^*), \quad \zeta_j^* = [-\omega_j^*\ p_j^{c,*}]^T, \quad j \in N. \tag{6.18e}$$

Proof of Lemma 6.1: In order to show that $\omega^* = \mathbf{0}_{|N|}$, we sum equations (6.6b) at equilibrium for all $j \in N$, resulting in $\sum_{j\in N} s_j^* = \sum_{j\in N} p_j^L$, which shows that $\sum_{j\in N} d_j^{u,*} = 0$ (by summing (6.18b) and (6.18c) over all $j \in G$ and $j \in L$ respectively). Then, Assumption 6.3 implies that this equality holds only if $\omega^* = \mathbf{0}_{|N|}$. ■

Proof of Theorem 6.1: Due to Assumption 6.1, C_j' and C_{dj}' are strictly increasing and hence invertible. Therefore all variables in (6.10) are well-defined. Also, Assumption 6.1 guarantees that the OGLC optimisation problem (6.7) is convex and has a continuously differentiable cost function. Thus, a point $(\bar{p}^M, \bar{d}^c)$ is a global minimum for (6.7) if and only if it satisfies the KKT conditions [104]

$$C_j'(\bar{p}_j^M) = \nu - \lambda_j^+ + \lambda_j^-,\ j \in G, \tag{6.19a}$$

$$C_{dj}'(\bar{d}_j^c) = -\nu - \mu_j^+ + \mu_j^-,\ j \in N, \tag{6.19b}$$

$$\sum_{j\in G} \bar{p}_j^M = \sum_{j\in N} (\bar{d}_j^c + p_j^L), \tag{6.19c}$$

$$p_j^{M,min} \le \bar{p}_j^M \le p_j^{M,max},\ j \in G, \tag{6.19d}$$

$$d_j^{c,min} \le \bar{d}_j^c \le d_j^{c,max},\ j \in N, \tag{6.19e}$$

$$\lambda_j^+(\bar{p}_j^M - p_j^{M,max}) = 0,\ \lambda_j^-(\bar{p}_j^M - p_j^{M,min}) = 0,\ j \in G, \tag{6.19f}$$

$$\mu^+(\bar{d}^c_j - d^{c,max}_j) = 0,\ \mu^-(\bar{d}^c_j - d^{c,min}_j) = 0,\ j \in N, \tag{6.19g}$$

for some constants $\nu \in \mathbb{R}$ and $\lambda^+_j, \lambda^-_j, \mu^+_j, \mu^-_j \geq 0$. It will be shown below that these conditions are satisfied by the equilibrium values $(\bar{p}^M, \bar{d}^c) = (p^{M,*}, d^{c,*})$ defined by equations (6.18d), (6.18e) and (6.10).

Since C'_j and C'_{dj} are strictly increasing, we can uniquely define $\beta^{M,max}_j := C'_j(p^{M,max}_j)$, $\beta^{M,min}_j := C'_j(p^{M,min}_j)$, $\beta^{c,max}_j := -C'_{dj}(d^{c,max}_j)$, and $\beta^{c,min}_j := -C'_{dj}(d^{c,min}_j)$. We let $\beta^*_0 = p^{c,*}_j$ and note that p^c_j are equal $\forall j$ at equilibrium, therefore β^*_0 is the same at each bus j. We now define in terms of these quantities the nonnegative constants

$$\begin{aligned}
\lambda^+_j &:= (\beta^*_0 - \beta^{M,max}_j)\, \mathbb{1}_{(\beta^*_0 \geq \beta^{M,max}_j)},\\
\lambda^-_j &:= (\beta^{M,min}_j - \beta^*_0)\, \mathbb{1}_{(\beta^*_0 \leq \beta^{M,min}_j)},\\
\mu^+_j &:= (\beta^{c,max}_j - \beta^*_0)\, \mathbb{1}_{(\beta^*_0 \leq \beta^{c,max}_j)},\\
\mu^-_j &:= (\beta^*_0 - \beta^{c,min}_j)\, \mathbb{1}_{(\beta^*_0 \geq \beta^{c,min}_j)}.
\end{aligned}$$

Then, since $(C'_j)^{-1}(\beta^*_0) \geq p^{M,max}_j \Leftrightarrow \beta^*_0 \geq \beta^{M,max}_j$, $(C'_j)^{-1}(\beta^*_0) \leq p^{M,min}_j \Leftrightarrow \beta^*_0 \leq \beta^{M,min}_j$, $(C'_{dj})^{-1}(-\beta^*_0) \geq d^{c,max}_j \Leftrightarrow \beta^*_0 \leq \beta^{c,max}_j$, and $(C'_{dj})^{-1}(-\beta^*_0) \leq d^{c,min}_j \Leftrightarrow \beta^*_0 \geq \beta^{c,min}_j$, it follows by (6.18d), (6.18e), and (6.10) that the complementary slackness conditions (6.19f) and (6.19g) are satisfied.

Now define $\nu = \beta^*_0$. Then, it follows that $(C'_j)^{-1}(\nu - \lambda^+_j + \lambda^-_j) = (C'_j)^{-1}\Big([\beta^*_0]^{\beta^{M,max}_j}_{\beta^{M,min}_j}\Big) = [(C'_j)^{-1}(\beta^*_0)]^{p^{M,max}_j}_{p^{M,min}_j} = p^{M,*}_j$, by the above definitions and equations (6.18d) and (6.10). Thus, the optimality condition (6.19a) holds. Analogously, $(C'_{dj})^{-1}(-\nu - \mu^+ + \mu^-) = (C'_{dj})^{-1}\Big([-\beta^*_0]^{-\beta^{c,max}_j}_{-\beta^{c,min}_j}\Big) = [(C'_{dj})^{-1}(-\beta^*_0)]^{d^{c,max}_j}_{d^{c,min}_j} = d^{c,*}_j$, by (6.18e) and (6.10), satisfying (6.19b).

Summing equations (6.18b) and (6.18c) over all $j \in G$ and $j \in L$ respectively and using the fact that $\sum_{j \in N} d^{u,*}_j = 0$ as shown in the proof of Lemma 6.1 shows that (6.19c) holds. Finally, the saturation constraints in (6.10) verify (6.19d) and (6.19e).

Hence, the values $(\bar{p}^M, \bar{d}^c) = (p^{M,*}, d^{c,*})$ satisfy the KKT conditions (6.19). Therefore, the equilibrium values $p^{M,*}$ and $d^{c,*}$ define a global minimum for (6.7). ■

Proof of Theorem 6.2: We will use the dynamics in (6.3)–(6.6) and the conditions of Assumption 6.5 to define a Lyapunov function for the system (6.3)–(6.6).

Firstly, let $V_F(\omega^G) = \frac{1}{2}\sum_{j \in G} M_j(\omega_j - \omega^*_j)^2$. The time-derivative of V_F along the trajectories of (6.3)–(6.4) is given by

$$\dot{V}_F = \sum_{j\in N} (\omega_j - \omega_j^*)\bigg(- p_j^L + s_j - d_j^u - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij}\bigg),$$

by substituting (6.3b) for $\dot{\omega}_j$ for $j \in G$ and adding extra terms for $j \in L$, which are equal to zero by (6.3c). Subtracting the product of $(\omega_j - \omega_j^*)$ with each term in (6.18b) and (6.18c), this becomes

$$\begin{aligned}\dot{V}_F =& \sum_{j\in N}\bigg((\omega_j - \omega_j^*)(s_j - s_j^*) + (\omega_j - \omega_j^*)(-d_j^u - (-d_j^{u,*}))\bigg) \\ &+ \sum_{(i,j)\in E} (p_{ij} - p_{ij}^*)(\omega_j - \omega_i),\end{aligned} \tag{6.20}$$

using the equilibrium condition (6.18a) for the final term.

Furthermore, let $V_C(p^c) = \frac{1}{2}\sum_{j\in N}\gamma_j (p_j^c - p_j^{c,*})^2$. Using (6.6b) the time derivative of V_C can be written as

$$\dot{V}_C = \sum_{j\in N}(p_j^c - p_j^{c,*})\Big((-s_j + s_j^*) - \sum_{k:j\to k}(\psi_{jk} - \psi_{jk}^*) + \sum_{i:i\to j}(\psi_{ij} - \psi_{ij}^*)\Big). \tag{6.21}$$

Additionally, define $V_P(\eta) = \sum_{(i,j)\in E} B_{ij}\int_{\eta_{ij}^*}^{\eta_{ij}} (\sin\theta - \sin\eta_{ij}^*)\, d\theta$. Using (6.3a) and (6.3d), the time-derivative is given by

$$\begin{aligned}\dot{V}_P =& \sum_{(i,j)\in E} B_{ij}(\sin\eta_{ij} - \sin\eta_{ij}^*)(\omega_i - \omega_j) \\ =& \sum_{(i,j)\in E} (p_{ij} - p_{ij}^*)(\omega_i - \omega_j).\end{aligned} \tag{6.22}$$

Finally, consider $V_\psi(\psi) = \frac{1}{2}\sum_{(i,j)\in\tilde{E}}\gamma_{ij}(\psi_{ij} - \psi_{ij}^*)^2$ with time derivative given by (6.6a) as

$$\dot{V}_\psi = \sum_{(i,j)\in\tilde{E}} (\psi_{ij} - \psi_{ij}^*)((p_i^c - p_i^{c,*}) - (p_j^c - p_j^{c,*})). \tag{6.23}$$

Furthermore, from the dissipativity condition in Assumption 6.5 the following holds: There exist open neighbourhoods U_j of ω_j^* and U_j^c of $p_j^{c,*}$ for each $j \in N$, open neighbourhoods X_j^G of $(x^{M,j,*}, x^{c,j,*}, x^{u,j,*})$ and X_j^L of $(x^{c,j,*}, x^{u,j,*})$ for each $j \in G$ and $j \in L$ respectively, and continuously differentiable, positive semidefinite functions $V_j^D(x^{M,j}, x^{c,j}, x^{u,j}), j \in G$ and $V_j^D(x^{c,j}, x^{u,j}), j \in L$, satisfying (6.8) with supply rate given by (6.9), i.e.,

$$\dot{V}_j^D \leq [(s_j - s_j^*)\ (-d_j^u - (-d_j^{u,*}))] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (\zeta_j - \zeta_j^*) - \phi_j(\zeta_j - \zeta_j^*),\ j \in N, \tag{6.24}$$

for all $\omega_j \in U_j$, p_j^c in U_j^c for $j \in N$ and all $(x^{M,j}, x^{c,j}, x^{u,j}) \in X_j^G$ and $(x^{c,j}, x^{u,j}) \in X_j^L$ for $j \in G$ and $j \in L$ respectively.

Based on the above, we define the function

$$V(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c) = V_F + V_P + \sum_{j \in N} V_j^D + V_C + V_\psi \tag{6.25}$$

which we aim to use in Lasalle's theorem. Using (6.20) - (6.23), the time derivative of V is given by

$$\dot{V} = \sum_{j \in N} \big[(\omega_j - \omega_j^*)(s_j - s_j^*) + \dot{V}_j^D + (p_j^c - p_j^{c,*})(-s_j + s_j^*) + (\omega_j - \omega_j^*)(-d_j^u - (-d_j^{u,*})\big]. \tag{6.26}$$

Using (6.24) it therefore holds that

$$\dot{V} \leq \sum_{j \in N} \Big(- \phi_j(\zeta_j - \zeta_j^*)\Big) \leq 0 \tag{6.27}$$

whenever $\omega_j \in U_j$, $p_j^c \in U_j^c$ for $j \in N$, $(x^{M,j}, x^{c,j}, x^{u,j}) \in X_j^G$ for $j \in G$, and $(x^{c,j}, x^{u,j}) \in X_j^L$ for $j \in L$.

Clearly V_F has a strict global minimum at $\omega^{G,*}$ and V_j^D has strict local minima at $(x^{M,j,*}, x^{c,j,*}, x^{u,j,*})$ and $(x^{c,j,*}, x^{u,j,*})$ for $j \in G$ and $j \in L$ respectively by Assumption 6.5 and Definition 6.2. Furthermore, V_C and V_ψ have strict global minima at $p^{c,*}$ and ψ^* respectively. Furthermore, Assumption 6.2 guarantees the existence of some neighbourhood of each η_{ij}^* in which V_P is increasing. Since the integrand is zero at the lower limit of the integration, η_{ij}^*, this immediately implies that V_P has a strict local minimum at η^*. Thus, V has a strict local minimum at the point $Q^* := (\eta^*, \psi^*, \omega^{G,*}, x^{M,*}, x^{c,*}, x^{u,*}, p^{c,*})$. From Assumption 6.4, we know that, provided $(\eta, \omega^G, x^c, x^u, p^c) \in T$, ω^L can be uniquely determined from these quantities. Therefore, the states of the differential equation system (6.3)–(6.6) with $(\eta, \omega^G, x^c, x^u, p^c)$ within the region T can be expressed as $(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c)$. We now choose a neighbourhood in the coordinates $(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c)$ about

Q^* on which the following hold:

1. Q^* is a strict minimum of V,
2. $(\eta, \omega^G, x^c, x^u, p^c) \in T$,
3. $\omega_j \in U_j$, $p_j^c \in U_j^c$ for $j \in N$, and $(x^{M,j}, x^{c,j}, x^{u,j}) \in X_j^G$, $(x^{c,j}, x^{u,j}) \in X_j^L$ for $j \in G$, $j \in L$ respectively[18],
4. $x^{M,j}$, $x^{c,j}$, and $x^{u,j}$ all lie within their respective neighbourhoods X_0 as defined in Section 6.3.1.

Recalling now (6.27), it is easy to see that within this neighbourhood, V is a nonincreasing function of all the system states and has a strict local minimum at Q^*. Consequently, the connected component of the level set $\{(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c) \colon V \leq \epsilon\}$ containing Q^* is guaranteed to be both compact and positively invariant with respect to the system (6.3)–(6.6) for sufficiently small $\epsilon > 0$. Therefore, there exists a compact positively invariant set Ξ for (6.3)–(6.6) containing Q^*.

Lasalle's Invariance Principle can now be applied with the function V on the compact positively invariant set Ξ. This guarantees that all solutions of (6.3)–(6.6) with initial conditions $(\eta(0), \psi(0), \omega^G(0), x^M(0), x^c(0), x^u(0), p^c(0)) \in \Xi$ converge to the largest invariant set within $\Xi \cap \{(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c) \colon \dot{V} = 0\}$. We now consider this invariant set. If $\dot{V} = 0$ holds at a point within Ξ, then (6.27) holds with equality, hence we must have $\omega = \omega^*$ and $p_j^c = p_j^{c,*}$ at all buses j where Assumption 6.5(a) holds. The fact that ω is constant guarantees from (6.3a), (6.3d) that η and p are also constant. This is sufficient to deduce from (6.3b)–(6.3c) that s is also constant. If instead Assumption 6.5(b) holds at a bus j we have that $\omega = \omega^*$ when $\dot{V} = 0$. Furthermore, we have the additional property that if ω_j and s_j are constant then p_j^c cannot be a sinusoid. This latter property guarantees that p_j^c is also constant by noting that the dynamics for the power command (6.6) with constant s_j, allow p_j^c to be either a constant or a sinusoid within a compact invariant set. Hence, we have $\omega = \omega^*$ and $p^c = p^{c,*}$ in the invariant set considered.

Furthermore, note that $\omega = \omega^*$, $p^c = p^{c,*}$ within the invariant set implies by the definitions in Section 6.2 that (x^M, x^c, x^u) converge to the point $(x^{M,*}, x^{c,*}, x^{u,*})$, at which V_j^D take strict local minima from Assumption 6.5. Thus, from (6.24) and (6.26) it follows that the values of V_j^D must decrease along all nontrivial trajectories

[18] This is possible because $\omega_j \in U_j$ for all $j \in L$ corresponds, by Assumption 6.4 and the continuity of the equations in (6.3)–(6.6), to requiring the states $(\eta, \omega^G, x^M, x^c, x^u, p^c)$ to lie in some open neighbourhood about Q^*.

within the invariant set, contradicting $\dot{V}_j^D = 0$. The fact that $(p^c, s) = (p^{c,*}, s^*)$ is sufficient to show that ψ equals some constant ψ^*. Using the same argument, it can be shown that within the invariant set, the fact that $\zeta = \zeta^*$ implies that $(x^M, x^c, x^u, p^M, d^c, d^u)$ converges to $(x^{M,*}, x^{c,*}, x^{u,*}, p^{M,*}, d^{c,*}, d^{u,*})$. Therefore, we conclude by Lasalle's Invariance Principle that all solutions of (6.3)–(6.6) with initial conditions $(\eta(0), \psi(0), \omega^G(0), x^M(0), x^c(0), x^u(0), p^c(0)) \in \Xi$ converge to the set of equilibrium points as defined in Definition 6.1. Finally, choosing for S any open neighbourhood of Q^* within Ξ completes the proof for convergence. From Lemma 6.1 it can then be deduced that $\omega^* = \mathbf{0}_{|N|}$. Furthermore, noting that all conditions of Theorem 6.1 hold shows the convergence to an optimal solution of the OGLC problem (6.7). ■

Remark 6.9 *It should be noted that for given $p^{c,*}$ and ω^* all $(\eta^*, x^{M,*}, x^{c,*}, x^{u,*})$ are unique. The uniqueness of η^* can be seen by noting that $\eta_{ij} = \theta_i - \theta_j, (i,j) \in E$, which requires η to lie in a space where a corresponding vector θ exists. Furthermore, the value of $p^{c,*}$ becomes unique when* (6.10) *holds. This follows from summing* (6.18b)–(6.18c) *over all buses and noting that the strict convexity of the cost functions and the monotonicity of f in* (6.10) *makes the static input output maps from $p^{c,*}$ to s^* monotonically increasing. The values of ψ^* are non-unique for general network topologies.*

Proof of Theorem 6.3: The proof follows from the fact that the function V_j^B defined as

$$V_j^B = \frac{1}{2} M_j (\omega_j - \omega_j^*)^2 + \frac{1}{2} \gamma_j (p_j^c - p_j^{c,*})^2 + V_j^D, \tag{6.28}$$

where V_j^D is as in (6.24) with $\phi_j = 0$, is a storage function for the system B_j. In particular, using arguments similar to those in the proof of Theorem 6.2, it can be shown that

$$\begin{aligned} \dot{V}_j^B \leq (p_j^c - p_j^{c,*}) \Bigg(\sum_{i:i\to j} (\psi_{ij} - \psi_{ij}^*) - \sum_{k:j\to k} (\psi_{jk} - \psi_{jk}^*) \Bigg) \\ + (-\omega_j - (-\omega_j^*)) \Bigg(\sum_{k:j\to k} (p_{jk} - p_{jk}^*) - \sum_{i:i\to j} (p_{ij} - p_{ij}^*) \Bigg) \end{aligned} \tag{6.29}$$

and therefore that system B_j is passive. ■

Proof of Lemma 6.2: Using (6.17d) at equilibrium, it can be deduced that $\chi_j^* = s_j^* - d_j^{u,*} - \sum_{k:j\to k} p_{jk}^* + \sum_{i:i\to j} p_{ij}^*$. Hence, it follows by summing (6.17b) at equilibrium over all buses that $\sum_{j \in N} s_j^* = \sum_{j \in N} \chi_j^* = \sum_{j \in N} s_j^* - d_j^{u,*}$, which results to $\sum_{j \in N} d_j^{u,*} = 0$. Hence, from Assumption 6.3, it follows that $\omega^* = \mathbf{0}_{|N|}$. ■

Proof of Proposition 6.1: We shall make use of the Lyapunov function in (6.25) to construct a new Lyapunov function for the system (6.3) – (6.5), (6.17).

First, consider the function

$$V_b(b, \chi, \omega) = \frac{1}{2} \sum_{j \in G} \big(M_j((b_j - b_j^*) - (\omega_j - \omega_j^*))^2 + \tau_{\chi,j}(\chi_j - \chi_j^*)^2 \big), \tag{6.30}$$

and note that its time-derivative along the trajectories of (6.17) is given by

$$\dot{V}_b = \sum_{j \in N} \Big(- (\chi_j - \chi_j^*)[(p_j^c - p_j^{c,*}) + (\chi_j - \chi_j^*)] \Big), \tag{6.31}$$

noting that for $j \in L$ it holds that $\chi = \chi^*$, and hence the added terms in (6.31) are equal to zero.

Furthermore, the time-derivative of $V_C(p^c) = \frac{1}{2} \sum_{j \in N} \gamma_j (p_j^c - p_j^{c,*})^2$ under (6.17b) is given by

$$\dot{V}_C = \sum_{j \in N} (p_j^c - p_j^{c,*}) \Big((-s_j + s_j^*) + (\chi_j - \chi_j^*) - \sum_{k: j \to k} (\psi_{jk} - \psi_{jk}^*) + \sum_{i: i \to j} (\psi_{ij} - \psi_{ij}^*) \Big). \tag{6.32}$$

Now consider the function V in (6.25) and note that its derivative is as in (6.26) with an extra term given by $\sum_{j \in N} (p_j^c - p_j^{c,*})(\chi_j - \chi_j^*)$. Then consider the function

$$V_O(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c, b, \chi) = V + V_b \tag{6.33}$$

which can be shown to have a time derivative given by

$$\dot{V}_O \leq \sum_{j \in N} \Big(- \phi_j(\zeta_j - \zeta_j^*) - (\chi_j - \chi_j^*)^2 \Big) \leq 0, \tag{6.34}$$

by similar arguments as in the proof of Theorem 6.2.

Now, in analogy to the proof of Theorem 6.2, it can be shown that an invariant compact set Ξ_O exists such that $\{(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c, b, \chi) \colon V_O \leq \epsilon\}$. Then, Lasalle's theorem can be invoked to show that all solutions of (6.3) – (6.5), (6.17) with initial conditions within Ξ_O will converge to the largest invariant set within $\Xi_O \cap \{(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c, b, \chi) \colon \dot{V} = 0\}$. Within this invariant set, it holds that

$(\omega, \chi) = (\omega^*, \chi^*)$. Applying the same arguments as in the proof of Theorem 6.2 shows that $(\eta, \psi, x^M, x^c, x^u, p^M, d^c, d^u, p^c)$ converges to $(\eta^*, \psi^*, x^{M,*}, x^{c,*}, x^{u,*}, p^{M,*}, d^{c,*}, d^{u,*}, p^{c,*})$ which implies the convergence of b to b^* from the dynamics in (6.17d). The optimality result follows directly from the proof of Theorem 6.1 since none of its arguments are affected from the dynamics in (6.17). ■

Appendix B

In this appendix we show that Assumption 6.5 is a necessary and sufficient condition for the passivity of bus systems B_j, described in Section 6.6.1, when their dynamics are affine nonlinear, i.e. are characterised by the following state space representation:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x). \end{aligned} \tag{6.35}$$

For the proof, we shall make use of Lemma 6.3 below. Within it, we shall consider the negative feedback interconnection of

$$\Sigma_1 : \begin{cases} \dot{x}_1 = u_1 \\ y_1 = h_1(x_1) \end{cases}, \Sigma_2 : \begin{cases} \dot{x}_2 = f_2(x_2) + g_2(x_2)u_2 \\ y_2 = h_2(x_2) + k_2(u_2) \end{cases} \tag{6.36}$$

such that $u_2 = y_1$ and $u_1 = r - y_2$, where $r(t) \in \mathbb{R}^n$ is some reference input applied to the closed-loop system, $x_1(t) \in \mathbb{R}^n, x_2(t) \in \mathbb{R}^{n_2}$ and $y_1(t), y_2(t) \in \mathbb{R}^n$ are the states and outputs of Σ_1 and Σ_2 respectively and $h_1, k_2 : \mathbb{R}^n \to \mathbb{R}^n$, and $f_2, g_2, h_2 : \mathbb{R}^{n_2} \to \mathbb{R}^n$ are functions describing the dynamics of Σ_1 and Σ_2. The closed-loop system, denoted by Σ, writes as

$$\Sigma : \begin{cases} \dot{x}_1 &= -h_2(x_2) - k_2(h_1(x_1)) + r, \\ \dot{x}_2 &= f_2(x_2) + g_2(x_2)h_1(x_1), \\ y_1 &= h_1(x_1). \end{cases} \tag{6.37}$$

Without loss of generality we also assume in the lemma below that $h_1(0), k_2(0), f_2(0), h_2(0)$ are equal to zero and the passivity properties stated are considered about this equilibrium point.

Lemma 6.3 *Consider the negative feedback interconnection Σ described by* (6.37) *of two subsystems Σ_1 and Σ_2 described by* (6.36)*. Assume that Σ is passive from r*

to y_1, and Σ_1 is passive from u_1 to y_1. Then, Σ_2 is passive from u_2 to y_2.

Proof of Lemma 6.3: From the passivity of Σ and [131, Corollary 4.1.5] there exists a positive definite continuously differentiable storage function $V(x_1, x_2)$, defined with respect to an equilibrium, such that

$$\begin{aligned}
&-\frac{\partial V}{\partial x_1}(x_1,x_2)h_2(x_2) - \frac{\partial V}{\partial x_1}(x_1,x_2)k_2(h_1(x_1)) \\
&+\frac{\partial V}{\partial x_2}(x_1,x_2)f_2(x_2) + \frac{\partial V}{\partial x_2}(x_1,x_2)g_2(x_2)h_1(x_1) \leq 0
\end{aligned} \tag{6.38}$$

and

$$\frac{\partial V}{\partial x_1}(x_1,x_2) = h_1^T(x_1).$$

Similarly, from the passivity of Σ_1 and [131, Corollary 4.1.5] there exists a positive definite continuously differentiable storage function V_1 such that

$$\frac{\partial V_1}{\partial x_1}(x_1) = h_1^T(x_1). \tag{6.39}$$

Hence,

$$\frac{\partial V}{\partial x_1}(x_1,x_2) = \frac{\partial V_1}{\partial x_1}(x_1). \tag{6.40}$$

Substituting this back to (6.38) yields

$$\begin{aligned}
&\frac{\partial V}{\partial x_2}(x_1,x_2)f_2(x_2) + \frac{\partial V}{\partial x_2}(x_1,x_2)g_2(x_2)h_1(x_1) \\
&\quad \leq \frac{\partial V_1}{\partial x_1}(x_1)(h_2(x_2) + k_2(h_1(x_1))) \\
&\quad = h_1^T(x_1)(h_2(x_2) + k_2(h_1(x_1))),
\end{aligned} \tag{6.41}$$

where the last inequality follows from (6.39). Now let V be written as

$$V(x_1,x_2) = V_1(x_1) + V_2(x_2) \tag{6.42}$$

for some continuously differentiable V_2. The fact that V_2 is only a function of x_2 descends from (6.40). Also note that $V_2(x_2)$ is positive definite. By substituting the above into (6.41), we conclude that

$$\frac{\partial V_2}{\partial x_2}(x_2)f_2(x_2) + \frac{\partial V_2}{\partial x_2}(x_2)g_2(x_2)u_2 \leq u_2^T y_2$$

which implies the passivity of Σ_2. ■

The following lemma shows that Assumption 6.5 is a necessary and sufficient condition for the passivity of generation bus system B_j. Note that the extension to load buses is trivial and thus omitted.

Lemma 6.4 *Consider the system described by* (6.3) - (6.6) *and its representation by systems I and B, defined in section 6.6.1 and let the dynamics for B_j be described by* (6.37). *Then, the dissipativity condition in Assumption 6.5 with $\phi = 0$ is necessary and sufficient for the passivity of subsystems $B_j, j \in G$ about the equilibrium point considered in Assumption 6.5.*

Proof of Lemma 6.4: The proof for the necessity of the condition follows from Lemma 6.3 when the following substitutions are made

$$r = \begin{bmatrix} \sum_{k:j\to k}(p_{jk} - p^*_{jk}) - \sum_{i:i\to j}(p_{ij} - p^*_{ij}) \\ \sum_{i:i\to j}(\psi_{ij} - \psi^*_{ij}) - \sum_{k:j\to k}(\psi_{jk} - \psi^*_{jk}) \end{bmatrix},$$

$$y_1 = \begin{bmatrix} -(\omega_j - \omega^*_j) \\ p^c_j - p^{c,*}_j \end{bmatrix},\ y_2 = \begin{bmatrix} (s_j - s^*_j) - (d^u_j - d^{u,*}_j) \\ (s_j - s^*_j) \end{bmatrix}.$$

The sufficiency proof follows directly from Theorem 6.3. ■

Chapter 7

Secondary frequency control with on-off load side participation in power networks

In this chapeter, we study the problem of secondary frequency regulation where ancillary services are provided via load-side participation. In particular, we consider on-off loads that switch when prescribed frequency thresholds are exceeded in order to assist existing secondary frequency control mechanisms. We show that system stability is not compromised despite the switching nature of the loads. However, such control policies are prone to Zeno-like behaviour, which limits the practicality of these schemes. As a remedy to this problem, we propose a hysteresis on-off policy and provide stability guarantees in this setting. We provide numerical investigations of the results on a realistic power network.

7.1 Introduction

As has been noted in the previous chapters, it is anticipated that renewable sources of generation will increase their penetration in power networks in the near future [106, 107]. This is expected to introduce intermittency in the power generated resulting in additional challenges in the real time operation of power networks that need to be addressed.

A main objective in the operation of a power system is to ensure that generation matches demand in real time. This is achieved by means of primary and secondary frequency control schemes with the latter also ensuring that the frequency returns to its nominal value (50Hz or 60Hz). Secondary frequency control is traditionally performed by having the generation side following demand [26]. However, a large penetration of renewable sources of generation limits the controllability of generation and at the same time makes the system more sensitive to disturbances due to the reduced system inertia [132]. Controllable loads are considered by many a promising solution to counterbalance intermittent generation, being able to adapt their demand based on frequency deviations, providing fast response at urgencies. Recently, various research studies focused on the inclusion of controllable demand to aid both primary control as in [42, 111] and secondary control as in [43, 44]. The material presented in Chapters 4–5 and 6 on primary and secondary control respectively fit within this context.

Further from providing ancillary services at urgencies, it is also desired that controllable loads are non-disruptive, i.e. their assistance should have a negligible effect on users comfort, see e.g. [74]. Non-disruptive load-side control schemes ensure that loads alter their demand at urgencies but return to their normal operation when the danger for the network has been surpassed. Moreover, in many occasions, a realistic representation of loads involves only a discrete set of possible demand values, e.g. on and off states. Hence, incorporating on-off controllable loads that appropriately react to frequency deviations in power networks is of particular interest in load-side participation schemes.

In this chapter, we consider controllable on-off loads that switch when some frequency deviation is reached so that they assist the network at urgencies (i.e. when large frequency deviations are experienced) and otherwise return to their original operation. It will be shown that the inclusion of such loads does not compromise the stability of the power network, and results in enhanced frequency performance. However, it will be observed that such controllable loads may switch arbitrarily fast within a finite interval of time, or in other words, exhibit Zeno behaviour. To avoid this, we propose on-off loads with hysteretic dynamics. Stability guarantees are again provided for this class of loads, and the absence of the Zeno phenomenon is analytically proven. We provide a numerical validation of our results through a simulation on the NPCC 140-bus system.

The structure of this chapter is as follows: Section 7.2 includes some basic notation and preliminaries and in Section 7.3 we present the power network model. In Section 7.4 we consider controllable demand that switches on/off whenever certain frequency thresholds are met and present our results concerning network stability. In Section 7.5, we consider controllable loads with hysteretic patterns and show that stability results extend to this case. Numerical investigations of the results are provided in Section 7.6. Finally, conclusions are drawn in Section 7.7.

7.2 Notation

Similarly to previous chapters, the notation within this chapter follows from Section 3.1. Moreover, we use $\mathbf{0}_n$ to denote $n \times 1$ vector with all elements equal to 0. In addition, the function $\mathrm{sgn}(x)$ takes a value of 1 when x is non-negative and -1 otherwise.

Within the chapter a class of switching systems will be considered and the notion of Filippov solutions, described in Section 3.3 will be used for their analysis (see also [92]).

In order to facilitate the analysis of differential equations with discontinuous vector fields $X : \mathbb{R}^n \to \mathbb{R}^n$, the dynamical system below (a differential inclusion) is often considered

$$\dot{x}(t) \in F[X](x(t)). \tag{7.1}$$

We remind that for systems described by (7.1), a Filippov solution is defined as an absolutely continuous map $x : [0, t_1] \to \mathbb{R}^n$ that satisfies (7.1) for almost all $t \in [0, t_1]$. For the system that will be studied in Section 7.4 we will show that Filippov solutions exist and are unique.

Remark 7.1 *Note that the use of Filippov solutions allows the study of systems with discontinuous dynamics when there are infinitely many switches at finite time, a phenomenon known as Zeno behaviour.*

7.3 Network model

We adopt the network description presented in Section 2.2.6 that has also been considered in the previous chapters. We describe the power network model by a connected graph (N, E) where $N = \{1, 2, \dots, |N|\}$ is the set of buses and $E \subseteq N \times N$ the set of transmission lines connecting the buses. Furthermore, we use (i, j) to denote the link connecting buses i and j and assume that the graph (N, E) is directed with arbitrary orientation, so that if $(i, j) \in E$ then $(j, i) \notin E$. For each $j \in N$, we use $i : i \to j$ and $k : j \to k$ to denote the sets of buses that are predecessors and successors of bus j respectively. It is important to note that the

form of the dynamics in (7.2)–(7.3) below is unaltered by any change in the graph ordering, and all of our results are independent of the choice of direction. The following assumptions are made for the network:
1) Bus voltage magnitudes are $|V_j| = 1$ p.u. for all $j \in N$.
2) Lines $(i, j) \in E$ are lossless and characterised by their susceptances $B_{ij} = B_{ji} > 0$.
3) Reactive power flows do not affect bus voltage phase angles and frequencies.

We use swing equations to describe the rate of change of frequency at generation buses, while power must be conserved at each of the load buses. This motivates the following system dynamics (e.g. [113]),

$$\dot{\eta}_{ij} = \omega_i - \omega_j,\ (i, j) \in E, \tag{7.2a}$$

$$M_j \dot{\omega}_j = -p_j^L + p_j^M - (d_j^c + d_j^u) - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij},\ j \in N, \tag{7.2b}$$

$$p_{ij} = B_{ij} \sin \eta_{ij} - p_{ij}^{nom},\ (i, j) \in E. \tag{7.2c}$$

In system (7.2) the time-dependent variables p_j^M, ω_j and d_j^c represent, respectively, deviations from a nominal value[1] of the mechanical power injection to the generator bus j, and the frequency and controllable load present at any bus j. The quantity d_j^u is also a time-dependent variable that represents the uncontrollable frequency-dependent load and generation damping present at bus j. Furthermore, the quantities η_{ij} and p_{ij} are time-dependent variables that represent, respectively, the power angle difference, and the deviation from the nominal value, p_{ij}^{nom}, of the power transmitted from bus i to bus j. The constant $M_j > 0$ denotes the generator inertia. We study the response of system (7.2) at a step change in the uncontrollable demand p_j^L at each bus j.

7.3.1 Generation and uncontrollable demand dynamics

We shall consider generation and uncontrollable demand dynamics described by

$$\dot{p}_j^M = -\alpha_j \omega_j,\ j \in N, \tag{7.3a}$$

$$d_j^u = A_j \omega_j,\ j \in N, \tag{7.3b}$$

[1]A nominal value is defined as an equilibrium of (7.2) with frequency equal to 50Hz (or 60Hz).

where $A_j > 0$ and $\alpha_j \geq 0$ for all $j \in N$. We assume that there exists at least one bus equipped with the integral controller above, i.e, $\max_{j \in N}(\alpha_j) > 0$. In case $\alpha_j = 0$, the generation output is equal to a constant, namely $p_j^M = p_j^{M,*}$.

Next, we will consider two classes of decentralised control schemes for discrete loads that provide ancillary services to the power network in the secondary control time-frame and investigate their performance and stability properties. As it will be discussed within the chapter, the discrete character of the loads leads to discontinuous system dynamics that can introduce additional complications that need to be explicitly addressed.

7.4 Loads with switching

7.4.1 Problem formulation

In this section, we consider frequency dependent on-off loads that respond to frequency deviations by switching to an appropriate state in order to aid the network at urgencies. As the network returns to its normal operating conditions, the loads return to their initial state as well, hence affecting users comfort for short periods only. In particular, for each[2] $j \in N$, we consider the following switching dynamics for the controllable loads:

$$d_j^c(\omega_j) = \begin{cases} \overline{d}_j, & \omega_j > \overline{\omega}_j, \\ 0, & \underline{\omega}_j < \omega_j \leq \overline{\omega}_j, \\ \underline{d}_j, & \omega_j \leq \underline{\omega}_j, \end{cases} \tag{7.4}$$

where $-\infty < \underline{d}_j \leq 0 \leq \overline{d}_j < +\infty$, and $\overline{\omega}_j > 0 > \underline{\omega}_j$. The dynamics in (7.4) are depicted on Figure 7.1. Note that these dynamics can be trivially extended to include more discrete values, that would possibly respond to higher frequency deviations.

Remark 7.2 *The dynamics in* (7.4) *demonstrate loads that switch OFF when frequency drops below some threshold value and ON when frequency grows above a different threshold value. Note that when* $\underline{d}_j$ *or* $\overline{d}_j$ *is zero, then this setting describes binary loads.*

[2]This can be trivially relaxed to any subset of N.

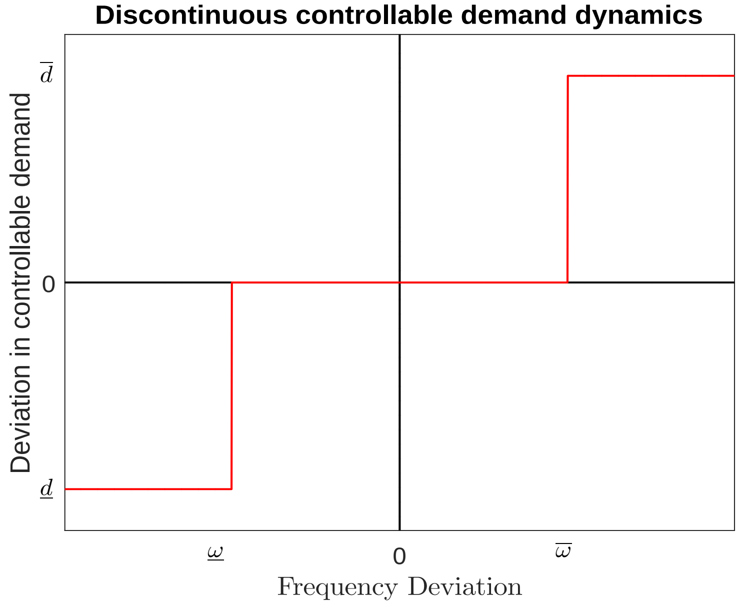

Figure 7.1. Switch dynamics for controllable loads as described by (7.4).

To cope with the switching dynamics of the loads, and to have well-defined solutions to system (7.4) for all time, we first define a Filippov set valued map as follows:

$$F[d_j^c] = \begin{cases} \{\overline{d}_j\}, & \omega_j > \overline{\omega}_j, \\ [0, \overline{d}_j], & \omega_j = \overline{\omega}_j \\ \{0\}, & \underline{\omega}_j < \omega_j < \overline{\omega}_j, \quad j \in N. \\ [\underline{d}_j, 0], & \omega_j = \underline{\omega}_j, \\ \{\underline{d}_j\}, & \omega_j < \underline{\omega}_j, \end{cases} \tag{7.5}$$

The state of the interconnected system (7.2)–(7.4) is denoted by $x = (\eta^T, \omega^T, (p^M)^T)^T$, where any variable without subscript represents a vector with all respective components. For a compact representation of the system, we consider the Filippov set valued map $Q : \mathbb{R}^n \to \mathcal{B}(\mathbb{R}^n)$, with $n = |E| + 2|N|$, and write the system dynamics as the following differential inclusion:

$$\dot{x} \in Q(x) \tag{7.6}$$

where

$$Q(x) = \begin{cases} \{\omega_i - \omega_j\},\ (i,j) \in E, \\ \Big\{ \frac{1}{M_j}(-p_j^L + p_j^M - A_j\omega_j - v_j - \sum_{k:j\to k} p_{jk} \\ + \sum_{i:i\to j} p_{ij}) : \quad v_j \in F[d_j^c] \Big\}, j \in N, \\ \{-\alpha_j \omega_j\},\ j \in N. \end{cases}$$

Remark 7.3 *As a result of the switching dynamics in* (7.4)*, the vector field in* (7.2b) *will become discontinuous. This discontinuity limits the applicability of classical solutions to the ordinary differential equations* (7.2)*, and asks for an appropriate notion of solutions. The most suitable notion depends on the objective and the problem at hand. Among several solution notions, see [133], we opt for Filippov solutions [14] which amounts to the relaxation of the differential equation to a differential inclusion, see* (7.6)*. The idea behind Filippov solutions is to study the behaviour of the vector field* around *a point of discontinuity, and consequently allow the vector field to take any value within an admissible set. As will be observed in Lemma 7.2, this does not spoil uniqueness of solutions for the dynamics considered in this chapter.*

7.4.2 Equilibria, existence and uniqueness of solutions

The discontinuous dynamics (7.4) introduce additional complexity in the analysis of the behaviour of (7.6). First, we study equilibria of the system, and then investigate existence and uniqueness of Filippov solutions. An equilibrium of (7.6) is defined as follows:

Definition 7.1 *The point $x^* = (\eta^*, \omega^*, p^{M,*})$ defines an equilibrium of the system* (7.6) *if $0_n \in Q(x^*)$.*

For an equilibrium of the system, the controllable demand takes its value from a set that depends on ω_j^*, i.e., $d_j^{c,*} \in F[d_j^c](\omega_j^*), j \in N$. Lemma 7.1 below, proven in the appendix, shows that this set is singleton, namely $Q(x^*) = \{0_n\}$, and $\omega^* = \mathbf{0}_{|N|} = d_j^{c,*}$.

Lemma 7.1 *For any equilibrium point $x^* = (\eta^*, \omega^*, p^{M,*})$ of* (7.6)*, we have $\omega^* = \mathbf{0}_{|N|}$ and $Q(x^*) = \{0_n\}$.*

It should further be noted that within the rest of the chapter the existence of some equilibrium of (7.6) is assumed. As evident from Lemma 7.1, the conditions for existence of an equilibrium can be studied independent of the switching loads, see e.g.[114].

In addition, we impose a constraint on the differences of the phase angles at the equilibrium. This assumption, stated below, is ubiquitous in power network literature, and is treated as a security constraint.

Assumption 7.1 $|\eta_{ij}^*| < \frac{\pi}{2}$ *for all* $(i,j) \in E$*.*

The following lemma, proven in the appendix, establishes existence and uniqueness of solutions to (7.2)–(7.4).

Lemma 7.2 *There exists a unique Filippov solution of* (7.2)–(7.4) *starting from any initial condition* $x_0 = (\eta(0), \omega(0), p^M(0)) \in \mathbb{R}^n$*.*

7.4.3 Stability

We now state the main result of this section, proven in the appendix:

Theorem 7.1 *Suppose that there exists an equilibrium* $(\eta^*, \omega^*, p^{M,*})$ *of* (7.6) *for which Assumption 7.1 is satisfied. Then there exists an open neighbourhood* Ξ *of this equilibrium such that Filippov solutions* (η, ω, p^M) *of* (7.6) *starting in this region asymptotically converge to the set of equilibria of the system. In particular, the frequency vector* ω *converges to* $\omega^* = 0_{|N|}$*.*

The theorem above establishes stability of the power network (7.2)-(7.3) with on-off load side control (7.4), and shows that frequency is restored to its nominal value after a transient load-side participation.

7.4.4 Zeno behaviour

A possible feature of switching and hybrid systems is the occurrence of infinitely many switches within some finite time, a phenomenon known as Zeno behaviour (e.g. [102]). Such behaviour is often undesirable and impedes practical implementations.

In our setting, Zeno behaviour may occur in on-off loads as shown numerically in Section 7.6. The reason such behaviour may occur is that the frequency derivative may change sign when passing a discontinuity, causing the vector field to point towards the discontinuity. For instance, suppose that $0 < \dot{\omega}_j(t_1) < \overline{d}_j$ for some time instant $t_1 > 0$, and that the threshold $\overline{\omega}_j$ is met at this time. Then, the load d_j^c switches on, causing a sign change in the value of $\dot{\omega}_j$. Hence, the frequency vector field will point at a direction of frequency decrease that will force the load to switch off. These on/off switches occur infinitely many times in a finite time, resulting in the aforementioned Zeno behaviour. Note that this phenomenon is only observed here during the transient response of the loads, as the mechanical power injection (7.3) will eventually dictate the sign of the vector field and regulate the frequency to its nominal value as shown in Theorem 7.1.

7.5 Hysteresis on controllable loads

7.5.1 Problem formulation

In this section, we propose the use of hysteretic dynamics in on-off controllable loads, which means that a controllable load switches on and off at different frequency thresholds. As will be observed, this modification will ensure that the system does not exhibit Zeno behaviour. For relevant applications of hysteric dynamics in ruling out chattering, Zeno behaviour, and other undesired features, see. e.g. [103, 134, 135].

We consider the following hysteretic dynamics for controllable loads:

$$d_j^c = \overline{d}_j \sigma_j \tag{7.7a}$$

$$\sigma_j(t^+) = \begin{cases} \mathrm{sgn}(\omega_j), & |\omega_j| \geq \omega_j^1 \\ 0, & |\omega_j| \leq \omega_j^0 \\ \sigma_j(t), & \omega_j^0 \leq |\omega_j| \leq \omega_j^1 \end{cases} \tag{7.7b}$$

where $j \in N$, $t^+ = \lim_{\epsilon \to 0}(t + \epsilon)$, and the frequency thresholds ω_j^0, ω_j^1, satisfy $\omega_j^1 > \omega_j^0 > 0$.

The dynamics described in (7.7) are depicted in Figure 7.2. Note that σ_j takes its value from the set $P = \{-1, 0, 1\}$.

Now the behaviour of the system (7.2),(7.3),(7.7), can be described by the states $z = (x, \sigma)$, where $x = (\eta, \omega, p^M) \in \mathbb{R}^n$ is the continuous state, and $\sigma \in P^{|N|}$ the discrete state. The domain of solution is then equal to $\mathbb{R}^n \times P^{|N|}$, which we denote in short by Λ.

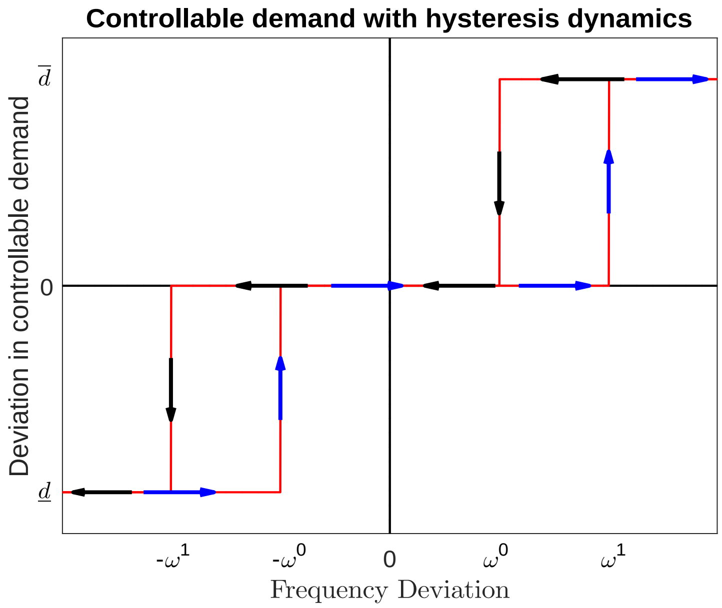

Figure 7.2. Hysteresis dynamics for controllable loads as described by (7.7).

The continuous part of the dynamics (7.2),(7.3),(7.7), is given by

$$\dot{\eta}_{ij} = \omega_i - \omega_j, \ (i,j) \in E, \tag{7.8a}$$

$$M_j \dot{\omega}_j = -p_j^L + p_j^M - (\bar{d}_j \sigma_j + A_j \omega_j) - \sum_{k:j \to k} p_{jk} + \sum_{i:i \to j} p_{ij}, \ j \in N, \tag{7.8b}$$

$$p_{ij} = B_{ij} \sin \eta_{ij} - p_{ij}^{nom}, \ (i,j) \in E, \tag{7.8c}$$

$$\dot{p}_j^M = -\alpha_j \omega_j, \ j \in N, \tag{7.8d}$$

$$\dot{\sigma}_j = 0, j \in N. \tag{7.8e}$$

This is valid when z belong to the set

$$C = \{z \in \Lambda : \sigma_j \in \mathcal{I}_j(\omega_j), \ \forall j \in N\} \tag{7.9}$$

where

$$\mathcal{I}_j(\omega_j) = \begin{cases} \{\mathrm{sgn}(\omega_j)\}, & |\omega_j| > \omega_j^1, \\ \{0\}, & |\omega_j| < \omega_j^0, \\ \{0, \mathrm{sgn}(\omega_j)\}, & \omega_j^0 \leq |\omega_j| \leq \omega_j^1. \end{cases}$$

Alternatively, when z belongs to the set D defined as

$$D = (\Lambda \setminus C) \cup \overline{D}, \tag{7.10}$$

where

$$\overline{D} = \{z \in \Lambda : |\omega_j| \in \{\omega_j^0, \omega_j^1\}, \sigma_j \in \mathcal{I}_j^D(\omega_j),\ \forall j \in N\}$$

and

$$\mathcal{I}_j^D(\omega_j) = \begin{cases} \{0\}, & |\omega_j| = \omega_j^1, \\ \{\mathrm{sgn}(\omega_j)\}, & |\omega_j| = \omega_j^0, \end{cases}$$

then, the system dynamics evolve according to the following discrete update rule:

$$\begin{aligned} x^+ &= x \\ \sigma_j(t^+) &= \begin{cases} \mathrm{sgn}(\omega_j), & |\omega_j| \geq \omega_j^1 \\ 0, & |\omega_j| \leq \omega_j^0 \end{cases}, j \in N, \end{aligned} \tag{7.11}$$

where the latter is in agreement with (7.7b). Furthermore, note that $C \cap D = \overline{D}$. We can now provide the following compact representation for the hybrid system (7.2),(7.3),(7.7),

$$\dot{z} = f(z), z \in C, \tag{7.12a}$$

$$z^+ = g(z), z \in D, \tag{7.12b}$$

where C and D are given by (7.9) and (7.10) respectively, and $z^+ = z(t^+)$. The maps $f(z) : C \to \Lambda$ and $g(z) : D \to C$ are given by (7.8) and (7.11), respectively. Note that $z^+ = g(z)$ represents a discrete dynamical system where z^+ is determined by the current value of the state z and the update rule given by g.

We assume that the initial conditions are compatible with (7.12), or essentially with the transition map in (7.7b). This means that, for each $j \in N$, $\sigma_j(0) \in \mathcal{I}_j(\omega_j(0))$. We write the condition above in vector form as $\sigma(0) \in \mathcal{I}(\omega(0))$, and we denote the set of compatible initial conditions by $\Lambda_0 \subseteq \Lambda$.

7.5.2 Analysis of equilibria and solutions

Before investigating stability of the hybrid system in (7.12), we characterise its equilibria, and establish existence and uniqueness of solutions.

Note that we call a point $z^* = (x^*, \sigma^*)$ an equilibrium of (7.12) if $f(z^*) = 0, z^* \in C$ or $z^* = g(z^*), z^* \in D$. Now, we state the following lemma:

Lemma 7.3 *For any equilibrium point $z^* = (x^*, \sigma^*)$ of (7.12), we have $\omega^* = \sigma^* = \mathbf{0}_{|N|}$. Moreover, $z^* \in C$.*

To proceed further, we need to assume existence of some equilibrium of (7.12). As evident from Lemma 7.3, the feasibility of this assumption is independent of the on-off loads and has been studied in literature (e.g. [114]).

We shall borrow the definitions of a hybrid time domain and solution to a hybrid system from [102, 103], also presented in Definition 3.11 in Section 3.4.

For convenience in the presentation we will refer to maximal solutions by just solutions. Existence of solutions to (7.12) are established in the following lemma.

Lemma 7.4 *There exists a complete solution $z = (x, \sigma)$ to (7.12), starting from any initial condition $z(0, 0) \in \Lambda_0$.*

Furthermore, the following proposition shows the existence of some finite dwell time between switches of states σ_j for any bounded solution. Within it, we denote the time-instants where the value of σ_j changes by $t_{\ell,j}, \ell \in \mathbb{N}_0, j \in N$.

Proposition 7.1 *For any complete bounded solution of (7.12), there exists $\tau_j > 0$ such that $\min_{\ell \geq 1}(t_{\ell+1,j} - t_{\ell,j}) \geq \tau_j$ for any $j \in N$.*

Remark 7.4 *The importance of Proposition 7.1 is that it shows that no Zeno behaviour will occur for any complete bounded solution of system (7.12). This is because for any finite time interval $\tau = \min_j \tau_j, j \in N$, the vector σ changes at most $|N|$ times. This highlights the practical advantage of (7.12) compared to (7.6). This analytic result is verified by numerical simulations in a realistic power network, as discussed in Section 7.6.*

7.5.3 Stability of hysteresis system

Now, we are at the position to state the stability result concerning the system (7.12).

Theorem 7.2 *Let* $z^* = (x^*, \sigma^*)$*, with* $\omega^* = \sigma^* = 0_{|N|}$*, be an equilibrium of* (7.12) *for which Assumption 7.1 holds. Then there exists an open neighbourhood* S *of* x^* *such that solutions* $z = (x, \sigma)$*, with* $x(0) \in S$ *and* $\sigma(0) \in \mathcal{I}(\omega(0))$*, asymptotically converge to the set of equilibria of* (7.12)*. In particular, the vectors* ω *and* σ *converge to the vector* $0_{|N|}$*.*

Remark 7.5 *Theorem 7.2 shows that the hysteretic dynamics in* (7.7) *do not compromise the stability of the system. This, together with the absence of Zeno behaviour shown in Proposition 7.1, promotes the use of hysteretic dynamics as a means to provide practical and non-disruptive on-off load side control to the power network.*

Remark 7.6 *It should be noted that none of the controllable loads considered in Sections 7.4 and 7.5 is active when the frequency is close to its nominal value. This allows various static optimality results such as the ones presented in Chapter 6 and [58] to be incorporated in this context.*

Remark 7.7 *Although the controllable loads do not participate at the equilibrium, they provide ancillary services to the network, and improve the performance in transient time. This is numerically investigated in Section 7.6. To clarify, note that the convergence region in Theorem 7.2 is not restricted by the switches, but rather by the nonlinearity of the frequency dynamics.*

7.6 Simulation on the NPCC 140-bus system

In this section we verify our analytic results with a numerical simulation on the Northeast Power Coordinating Council (NPCC) 140-bus interconnection system, using the Power System Toolbox [115]. This model is more detailed and realistic than our analytical one, including line resistances, a DC12 exciter model, a subtransient reactance generator model, and turbine governor dynamics[3].

The test system consists of 93 load buses serving different types of loads including constant active and reactive loads and 47 generation buses. The overall system has a total real power of 28.55GW. For our simulation, we added three uncontrollable

[3]The details of the simulation models can be found in the Power System Toolbox data file datanp48.

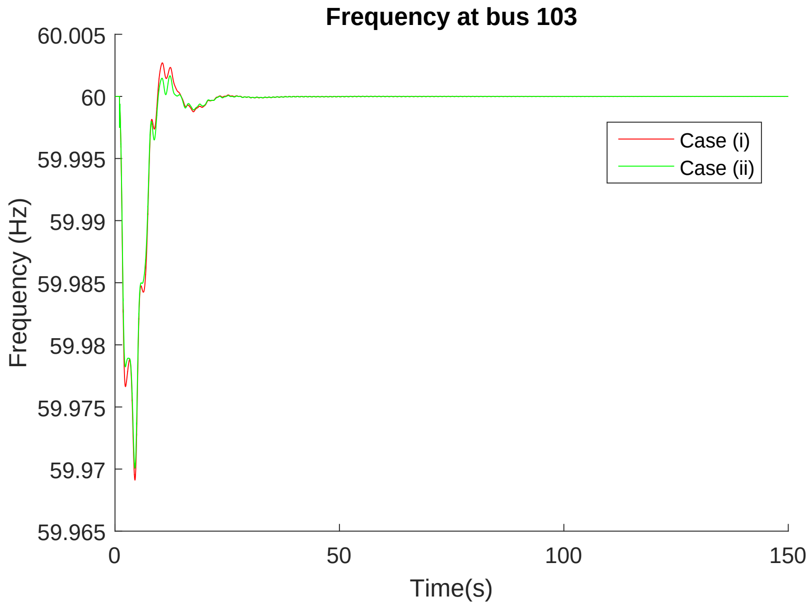

Figure 7.3. Frequency at bus 103 with controllable load dynamics as in the following two cases: i) Switching case, ii) Hysteresis case.

loads on units 2, 8, 9, 16 and 17, each having a step increase of magnitude 3 p.u. (base 100MVA) at $t = 1$ second.

Controllable demand was considered within the simulations, with controllable loads controlled every 10ms. Additionally, generators were considered at all generation buses, with dynamics as described by (7.3a).

The system was tested at two different cases. In case (i), on-off controllable loads with dynamics as in (7.4) were included at 20 load buses. The values for $\overline{\omega}_j$ were selected from a uniform distribution within the range [0.02 0.07] and those of $\underline{\omega}_j$ by $\underline{\omega}_j = -\overline{\omega}_j$. Controllable loads were also included at 20 load buses for case (ii), but with dynamics described by (7.7). To have a fair comparison, the same frequency thresholds were used for both cases, with $\omega_j^1 = \overline{\omega}_j$ and $\omega_j^0 = \omega_j^1/4$. Also, $\overline{d} = -\underline{d} = 0.2p.u.$ was used for both cases. Cases (i) and (ii) will be referred as the 'switching' and 'hysteresis' cases respectively.

The frequency at bus 103 for the two tested cases is shown in Fig. 7.3, where it can be seen that frequency returns to its nominal value for both cases, as suggested in Theorems 7.1 and 7.2. Moreover, Fig. 7.4 demonstrates that the inclusion of switching loads decreases the maximum overshoot in frequency, by comparing the largest deviation in frequency with and without on-off controllable loads at buses $1-40$, where frequency overshoot was seen to be the largest. Figure 7.5 shows controllable demand at 4 buses for case (i), demonstrating very fast switches and

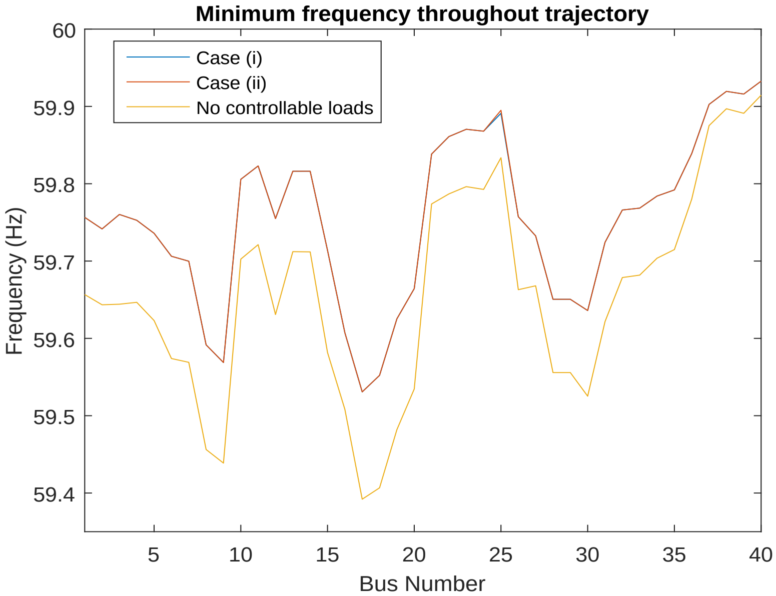

Figure 7.4. Largest frequency overshoot for buses $1-40$ for three cases: (i) Use of switching loads, (ii) Use of hysteresis loads, (iii) No use of controllable loads. Note that the graphs for cases (i), (ii) are almost identical and indistinguishable in the figure.

indicating Zeno behaviour. In contrast, when case (ii) is considered, such fast switching in loads is not observed on those 4 buses, as exhibited in Figure 7.6. Note that the 4 demonstrated buses where selected to be those with the minimum time between consecutive switches in the hysteretic loads of case (ii). Furthermore, both Figures 7.5 and 7.6 demonstrate times up to $10s$ since all loads stayed switched off afterwards. The latter demonstrates the non-disruptive nature of the two schemes, since loads return to their nominal demand after a brief period. This numerical investigation supports the analysis of this chapter, verifying that hysteresis eliminates Zeno behaviour at controllable loads.

7.7 Conclusion

We have considered the problem of secondary frequency control where controllable on-off loads provide ancillary services. We first considered loads that switch on when some frequency threshold is reached and off otherwise. Stability guarantees are provided for such loads. Furthermore, it is discussed that such schemes might exhibit arbitrarily fast switching, which might limit their practicality. To cope with this issue, on-off loads with hysteretic dynamics were considered. It has been shown that such loads do not exhibit any Zeno behaviour and that their inclusion

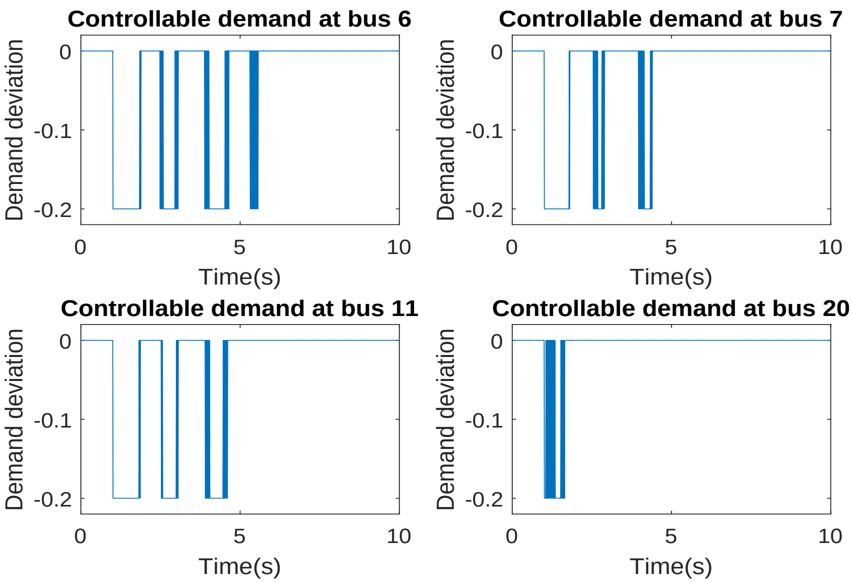

Figure 7.5. Controllable demand deviations at 4 buses with Switching on-off loads.

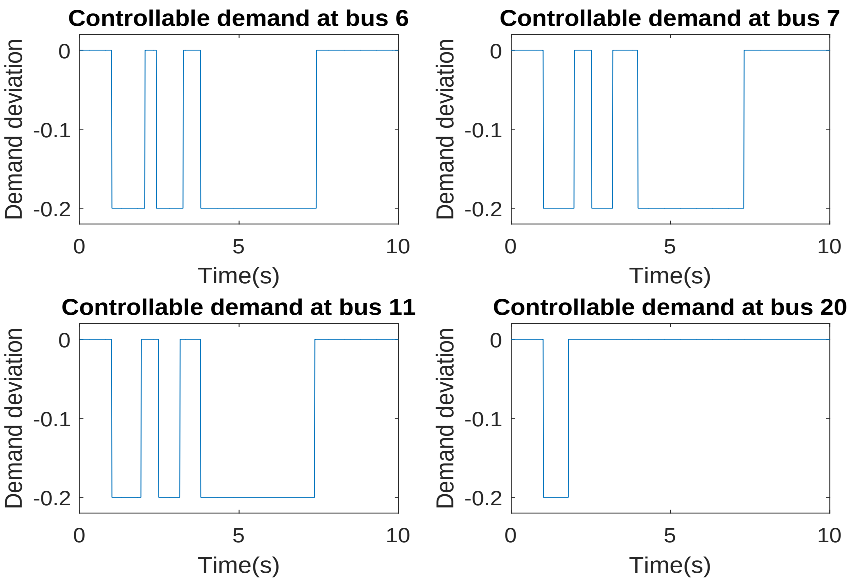

Figure 7.6. Controllable demand deviations at 4 buses with Hysteresis on-off loads.

does not compromise power network stability. Hence, such schemes are usable for practical implementations. Both schemes ensure that controllable loads return to their nominal behaviour at equilibrium and hence that disruptions occur for brief periods only. Our analytic results have been verified with numerical simulations on the NPCC 140-bus system where it was shown that the presence of on-off loads reduces the frequency overshoot and that hysteresis schemes resolve issues caused by Zeno-like behaviour.

Appendix

This appendix contains the proofs of all lemmas and theorems of this chapter. Within the proofs of Lemma 7.1 and Theorem 7.1 we will make use of the following equilibrium equations for system (7.2)–(7.3), which follow from Definition 7.1 and Lemma 7.1. Within the appendix, we let $\bar{N}$ be the set of all buses with non-trivial generation dynamics, i.e. those where $\alpha_j > 0$ in (7.3a).

$$0 = \omega_i^* - \omega_j^*,\ (i,j) \in E, \tag{7.13a}$$

$$0 = -p_j^L + p_j^{M,*} - \sum_{k:j\to k} p_{jk}^* + \sum_{i:i\to j} p_{ij}^*,\ j \in N, \tag{7.13b}$$

$$p_{ij}^* = B_{ij}\sin\eta_{ij}^* - p_{ij}^{nom},\ (i,j) \in E, \tag{7.13c}$$

$$0 = \omega_j^*, j \in \bar{N}, \tag{7.13d}$$

$$0 = d_j^{c,*}, j \in N. \tag{7.13e}$$

Proof of Lemma 7.1: The result follows from equilibrium equations (7.13a) and (7.13d). Note that $\omega^* = \mathbf{0}_{|N|}$ implies that $d^{u,*} = d^{c,*} = \mathbf{0}_{|N|}$ from (7.3b) and (7.4) respectively. ■

Proof of Lemma 7.2: Noting that the vector field is piecewise continuous, we use Proposition 5 in [92] to establish existence and uniqueness of solutions. To this end, observe that at any point of discontinuity one of the following holds:

(i) The vector fields point in the same direction.

(ii) The vector fields point towards the discontinuity.

In fact, at a point of discontinuity, say $\omega_j = \overline{\omega}_j$, we have $\dot{\omega}|_{d_j^c=\bar{d}_j} \leq \dot{\omega}|_{d_j^c=0}$. This rules out the case where the vector field points away from the discontinuity from both sides. An analogous argument can be made for the case $\omega_j = \underline{\omega}_j$. Consequently, whenever the solution reaches a point of discontinuity it will either continue in the same direction (as in (i)) or stay there (as in (ii)) and therefore existence and uniqueness of solutions follow from [92, Prop. 5]. ■

Proof of Theorem 7.1: We will use the dynamics in (7.2) and (7.3) to define a Lyapunov function for system (7.2)–(7.4).

First, consider $V_F(\omega) = \frac{1}{2}\sum_{j\in N} M_j\omega_j^2$. By substituting (7.2b) for $\dot{\omega}_j$ and using the differential inclusion for d_j^c for $j \in N$, the time-derivative along solutions of (7.2)–

(7.4) is then obtained as

$$\dot{V}_F = \left\{ \sum_{j\in N} \omega_j(-p_j^L + p_j^M - v_j - d_j^u - \sum_{k:j\to k} p_{jk} + \sum_{i:i\to j} p_{ij}) : \;\; v_j \in F[d_j^c(\omega_j)] \right\}.$$

Here, $\dot{V}_F$ has to be interpreted as the set-valued derivative of V_F with respect to (7.2)–(7.4). By (7.13), it is easy to observe that

$$\begin{aligned} \dot{V}_F = \Bigg\{ & \sum_{(i,j)\in E} (p_{ij} - p_{ij}^*)(\omega_j - \omega_i) \\ & + \sum_{j\in N} \omega_j (p_j^M - p_j^{M,*} - v_j - d_j^u) : \;\; v_j \in F[d_j^c(\omega_j)] \Bigg\}. \end{aligned} \tag{7.14}$$

Additionally, consider $V_P(\eta) = \sum_{(i,j)\in E} B_{ij} \int_{\eta_{ij}^*}^{\eta_{ij}} (\sin\phi - \sin\eta_{ij}^*)\, d\phi$. Using (7.2a) and (7.2c), the time-derivative equals

$$\begin{aligned} \dot{V}_P &= \sum_{(i,j)\in E} B_{ij}(\sin\eta_{ij} - \sin\eta_{ij}^*)(\omega_i - \omega_j) \\ &= \sum_{(i,j)\in E} (p_{ij} - p_{ij}^*)(\omega_i - \omega_j). \end{aligned} \tag{7.15}$$

Finally, consider the function $V_M(p^M) = \frac{1}{2}\sum_{j\in N}(p_j^M - p_j^{M,*})^2$. Using (7.3a), its time derivative is given by

$$\dot{V}_M = \sum_{j\in N} (p_j^M - p_j^{M,*})(-\omega_j), \tag{7.16}$$

noting that $p_j^M = p_j^{M,*}$ for all $j \in N/\bar{N}$. Based on the above, we define the function

$$V(\eta, \omega, p^M) = V_F(\omega) + V_P(\eta) + V_M(p^M). \tag{7.17}$$

By (7.3b) and (7.14)–(7.16), we have

$$\dot{V} = \left\{ \sum_{j\in N} (-\omega_j v_j - A_j(\omega_j)^2) : \;\; v_j \in F[d_j^c(\omega_j)] \right\}. \tag{7.18}$$

Using (7.5), we conclude that

$$\max \dot{V} \le -\sum_{j\in N} A_j(\omega_j)^2 \le 0, \tag{7.19}$$

where the maximum is taken over all the points in the set given by the right hand side of (7.18).

Clearly V_F and V_M have strict global minima at $\omega = \omega^* = 0$ and $p^M = p^{M,*}$ respectively. Additionally, Assumption 7.1 guarantees the existence of some neighbourhood of each η^*_{ij} on which the respective integrand in the definition of V_P is increasing. Since the integrand is zero at the lower limit, η^*_{ij}, this immediately implies that V_P has a strict local minimum at η^*. Thus, V has a strict local minimum at the point $x^* := (\eta^*, \omega^*, p^{M,*})$. We can thus choose a neighbourhood in the coordinates (η, ω, p^M) about x^* which is a strict minimum of V. From (7.19), within this neighbourhood, V is a non-increasing function of all states and has a strict local minimum at x^*. Consequently, the connected component of the level set $\{(\eta, \omega, p^M) \colon V \leq \epsilon\}$ containing x^* is both compact and positively invariant with respect to (7.2)–(7.4) for all sufficiently small $\epsilon > 0$. Hence, there exists a compact positively invariant set Ξ for (7.2)–(7.4) containing x^*.

Therefore, Theorem 3 in [97] can be invoked for the function V on the compact and positively invariant set Ξ along solutions of (7.6). This guarantees that all solutions of (7.2)–(7.4) with initial conditions $(\eta(0), \omega(0), p^M(0)) \in \Xi$ converge to the largest invariant set within $\Xi \cap \{(\eta, \omega, p^M) \colon 0 \in \dot{V}\}$. Note that this invariant set satisfies $\omega^* = \mathbf{0}_{|N|}$ and hence $F[d^c_j]$ is single valued from (7.5). Consequently, $\dot{V} = \{0\}$. On the invariant set, (7.19) holds with equality, hence we must have $\omega = \omega^* = \mathbf{0}_{|N|}$ where the latter follows from Lemma 7.1, as well as $d^{c,*} = d^{u,*} = \mathbf{0}_{|N|}$. This suggests from (7.2a) and (7.3a) that the vectors η and p^M are equal to some constant vectors $\bar{\eta}$ and $\bar{p}^M$, on the invariant set . Therefore, we conclude by [97, Thm. 3] that all Filippov solutions of (7.2)–(7.4) with initial conditions $(\eta(0), \omega(0), p^M(0)) \in \Xi$ converge to the set of equilibria defined in Definition 7.1. This completes the proof. ■

Proof of Lemma 7.3: Recall that any equilibrium z^* of (7.12) satisfies $f(z^*) = 0, z^* \in C$ or $g(z^*) = z^*, z^* \in D$. The latter case is excluded since $g(z) : D \to C$. Therefore, $z^* \in C$. From equations (7.8a) and (7.8d) at equilibrium, it follows that $\omega^* = \mathbf{0}_{|N|}$, which implies that $\sigma_j = \mathbf{0}_{|N|}$. ■

Proof of Lemma 7.4: To show the existence of solutions, first note that for any initial condition it holds that either $z(0,0) \in C \setminus \overline{D}$ or $z(0,0) \in D$. The latter results in $z(0,1) \in C$ as $g(z) : D \to C$. Then, from the continuity in the dynamics in (7.8), it follows that a solution exists for t sufficiently close to 0. Given $z(0,\ell) \in C$ for $\ell = \{0,1\}$, let $\bar{t} > 0$ be the minimal time such that the solution remains within C. If $\bar{t} = \infty$ then we have concluded the argument. Otherwise there exists $\tau \geq \bar{t}$ such that the solution exists within $T_\ell = [0, \tau)$ and it holds that $z(\tau, \ell + 1) = g(z(\tau, \ell))$, where $z(\tau, \ell) \in D$ and $z(\tau, \ell + 1) \in C$. After this transition the solution can be extended starting from $z(\tau, \ell + 1) \in C$ by repeating the above argument.

To show that any maximal solution z is complete, first let t_ℓ be the time instant where the ℓ^{th} transition occurs, and then consider the time domain K of z and assume that K is bounded. Then there exist finitely many intervals of the form $[t_\ell, t_{\ell+1}] \times \{\ell\}$ with the last interval either $[t_\ell, t_{\ell+1}] \times \{\ell\}$ or $[t_\ell, t_{\ell+1}) \times \{\ell\}$ with $t_{\ell+1} < \infty$. From the Lipschitz continuity of the vector field f we have, however, that if no discrete transition occurs in the last interval the solution is defined for all $t > t_\ell$. Thus, since $t_{\ell+1} < \infty$ we have that a transition occurs at $t_{\ell+1}$ and therefore the solution z is not maximal. Hence, all maximal solutions are complete by contradiction. ■

Proof of Proposition 7.1: Consider any bounded solution of the the system (7.12) with states $z = (\eta, \omega, p^M, \sigma)$ and define $\epsilon_j = \omega_j^1 - \omega_j^0$ following the description in (7.7). For any finite time interval between two consecutive switches at bus j, i.e. $[t_{\ell,j}, t_{\ell+1,j}]$, the value of $\dot{\omega}_j$ is bounded from above by a constant, say $d\omega_j^{\max}$. The fact that $d\omega_j^{\max}$ is finite follows from boundedness of the solution. Then, from the continuity in ω_j, it follows that $t_{\ell+1,j} - t_{\ell,j} \geq \epsilon_j / d\omega_j^{\max}$. Notice that the condition provided in the lemma is stated from the second switching time to include the case $z(0) \in D$. ■

Proof of Theorem 7.2: For the proof we shall make use of the function V, described by (7.17). Using similar arguments as in the proof of Theorem 7.1 and defining $T_c = \{t : (t, \ell) \in K, z(t, \ell) \in C\}, T_d = \{t : (t, \ell) \in K, z(t, \ell) \in D\}$ it follows that

$$\dot{V} \leq -\sum_{j \in N} A_j (\omega_j)^2, t \in T_c \tag{7.20a}$$

$$V(g(z)) - V(z) = 0, t \in T_d, \tag{7.20b}$$

along any solution of (7.12). Note that when $z \in D$, the value of V remains constant as it only depends on x that is constant from (7.11).

Note that V is a function of x only, and is nonnegative for all x in a neighbourhood of the equilibrium x^*. Moreover, $V = 0$ yields $\omega = \omega^* = 0$, and thus $\sigma = \sigma^* = 0$. Hence, the function V serves as a Lyapunov function for the hybrid system (7.12a). Then there exists a compact and positively invariant set $S = \{(x, \sigma) : x \in \Xi \text{ and } \sigma \in \mathcal{I}(\omega)\}$ for some neighbourhood Ξ of x^*. Note that the positively invariant set Ξ is obtained in the same vein as in the proof of Theorem 7.1, and that the set $\{(x, \sigma) : \sigma \in \mathcal{I}(\omega)\}$ is positively invariant by construction. Recall that, by Lemma 7.4 and Proposition 7.1, solutions of (7.12a) are complete, and the time interval between any two consecutive switches of individual loads is bounded from below by a positive number. Therefore, by [101, 102], there exists $r > 0$ such that solutions to (7.12) converge to the largest (weakly) invariant subset[4] of the set $\{z : V(z) = r\} \cap \{z \in C : \dot{V} = 0\} \cap S$. The characterisation of this invariant set follows in a similar way as in the proof of Theorem 7.1, noting that the equilibria of (7.12) are as described by Lemma 7.3. ■

[4]We use the notion of invariant sets provided in [101].

Conclusions

Chapter 8

Conclusions

This chapter concludes this thesis by summarising its main contributions and providing suggestions and ideas for further future research.

8.1 Summary of contribution

We have studied power network behaviour and derived conditions for the design of distributed schemes for generation and demand such that stability and optimality are achieved within the primary and secondary frequency control timeframes. For our results, we used tools from non-linear analysis, Lyapunov theory, passivity analysis, optimisation, and analysis of discontinuous and hybrid systems.

The main contributions of this thesis are the following:

1. We developed a framework for the design of decentralised generation and demand schemes within the primary frequency control timeframe such that stability and optimality guarantees are provided. The proposed framework incorporates a large class of dynamics, including the highly relevant case of high order dynamics, and allows for relaxed stability conditions compared to literature.

2. We have demonstrated optimality guarantees for frequency dependent loads with dead-band dynamics that provide ancillary service in primary frequency control. The additional complexities that follow from the discontinuous vector field derivatives were resolved by employing subgradient techniques.

3. We proposed a framework that extends the literature on the allowable continuous distributed generation and demand dynamics such that stability and

optimality guarantees are provided within secondary frequency control. Moreover, we have relaxed the requirements of the considered optimality scheme by making use of an appropriate observer.

4. We have shown that the incorporation of frequency dependent on-off loads as ancillary service does not compromise the stability of the power network. Furthermore, we demonstrated that when the on and off frequencies are equal then loads are prone to arbitrarily fast switching which is impractical. We resolved this problem by proposing hysteretic loads and shown that stability guarantees are retained within this setting.

These results are presented in Chapters 4–7. Below, we summarise the four main contribution chapters, providing appropriate interpretation to the presented analysis and results.

In Chapter 4 we considered the problem of designing decentralised schemes such that stability and optimality are achieved within the primary frequency control timeframe. We considered a strictly convex optimisation problem which ensured fairness in power allocation and balance between generation and demand. Moreover, we imposed a passivity condition, described by Assumption 4.2, on aggregate power supply variables at each bus. The main result, Theorem 4.2, ensures convergence of solutions to a global minimum of the considered optimisation problem (4.8). Furthermore, Theorem 4.4 shows that the inclusion of controllable loads results to a smaller steady state frequency deviation aiding in secondary frequency control. The analysis in this chapter allows for relaxed stability conditions and permits the inclusion of higher order systems which are of high relevance in power systems literature, as demonstrated with realistic examples from the NPCC network in Section 4.6. Also, in Proposition 4.1 and Corollary 4.1, we demonstrate how the results apply to non-linear systems and provide relaxed conditions compared to literature.

Chapter 5 considers the problem of designing distributed generation and demand schemes to provide ancillary service in the primary frequency control timeframe. In such schemes, loads respond to frequency only when some threshold is reached in order to support the network at urgencies. Furthermore, bounds for the maximum and minimum generation and demand are considered. The described schemes result to vector fields with discontinuous derivatives which impose additional complexities in the optimality analysis. To overcome these complexities and provide optimality guarantees we have employed subgradient techniques. Furthermore, appropriate stability results are stated based on the analysis of Chapter 4. The main result in this chapter, Theorem 5.2, demonstrates the stability and optimality of the considered schemes for arbitrary interconnections.

In both Chapters 4 and 5 the analytic results presented have been verified with realistic simulations on the IEEE New York/New England 68-bus interconnection system.

In Chapter 6 we studied the problem of designing distributed schemes for secondary frequency regulation in power networks such that stability and an economically optimal allocation are ensured. Since frequency is required to take its nominal value at equilibrium, as an objective of secondary frequency control, a different synchronising variable had to be used for optimality. For this purpose, we adopted an optimality scheme, described by (6.6), that allowed a locally communicated variable to be synchronised and guaranteed that the equilibrium frequency would be equal to the nominal. A strictly convex optimisation problem was posed which ensured generation/demand balance and frequency restoration along its solutions. Moreover, a decentralised dissipativity condition described by Assumption 6.5 was imposed on power supply variables to ensure convergence of solutions to a global minimum of the constructed optimisation problem, as demonstrated in Theorem 6.2. The proposed framework allowed to incorporate a broad range of generation and demand dynamics including the highly relevant higher order dynamics and is easily verifiable in linear systems by an appropriate LMI condition. Furthermore, we showed how the addition of an appropriate observer allowed to relax the requirement of the considered optimality scheme to have knowledge of uncontrollable frequency independent demand which is in many cases impractical, without compromising any of the demonstrated stability and optimality results. It is also explained how Assumption 6.5 is necessary and sufficient for the passivity of aggregate bus dynamics when the power supply dynamics are described by general affine nonlinear dynamics, providing further intuition on our approach. Our analysis is verified with simulations on the NPCC network which demonstrate the practicality of the proposed results.

Finally, in Chapter 7 we studied the effect of incorporating frequency dependent on/off loads as ancillary service to power networks within the secondary frequency control timeframe. An on/off model can in many cases be more realistic than a continuous one to describe load behaviour and its study is therefore relevant in power literature. Firstly, we considered loads that switched on and off when prescribed frequency thresholds were met. To study the dynamic behaviour of such loads we have made use of Filippov solutions, of which we have shown convergence in Theorem 7.1. However, it was seen that arbitrarily fast switching phenomena were possible in the transient behaviour of this system, due to vector field sign changes around the points of discontinuity. As a solution to this problem we considered loads that switched on and off at different frequencies, exhibiting hysteretic behaviour. For such loads, we have shown in Proposition 7.1 that no Zeno behaviour may be exhibited. Moreover, the convergence of hybrid solutions of this system is demonstrated by Theorem 7.2. Hence, it was shown that the considered switching loads do not compromise the stability of the power network. Their use as ancillary service is demonstrated by simulations on the NPCC 140 bus system where it is shown that the presence of switching loads significantly decreases frequency overshoot.

8.2 Future research directions

The analysis presented in this thesis motivates further research in order to address various questions that naturally arise from its study. Below, we discuss various relevant ideas and suggestions to extend this work.

Chapter 7 studies the stability of power networks in the presence of on/off frequency dependent loads. Further from switching loads, it considers continuous power supply consisting of generators integrating frequency and static uncontrollable loads. One potential extension in this analysis is to explore what classes of continuous dynamics allow to retain the presented stability guarantees when the considered on/off loads are present. The analysis in Chapter 6 may be useful in this sense, providing intuition about how to construct a framework such that the stability properties are retained when a broad range of continuous dynamics is considered. It is worth noting that any optimality interpretation on the considered continuous dynamics will not be affected by the presence of switching loads since those remain switched off at equilibrium.

It would be interesting to extend the analysis presented in Chapter 7 by considering on/off loads within primary frequency control. This is a highly relevant problem since ancillary services from on/off loads should be provided within a fast timeframe which coincides with that of primary frequency control. Such study might borrow from the analysis presented in Chapter 4 to consider continuous dynamics that allow for stability guarantees at the presence of switching loads. However, this proposed research direction introduces various technical difficulties, mainly resulting from frequency being different than nominal at equilibrium. These challenges become apparent when hybrid loads are considered where issues such as the presence of limit cycles and the existence of equilibria need to be addressed. Moreover, an optimality interpretation of switching loads would be sensible within primary control, considering how loads might be designed in order to minimise costs. However, this would possibly result to an integer programming optimisation problem which cannot in general be solved in polynomial time.

An interesting extension to Chapter 7 would also be to consider the case where loads switch when a particular event happens. The analysis in Chapter 7 can be seen as a special case of this, since the considered event is the reach of a particular frequency threshold at each bus. Such analysis would allow to include more exotic schemes that would possibly improve the performance of the network. One such example would be to design loads that switch when the maximum frequency deviation within local neighbouring buses reaches some threshold value, which would possibly result to a faster response at urgencies compared to the case of local frequency dependent switching thresholds. This analysis could be further extended to include varying switching thresholds, adapting according to the networks needs and possibly taking economic considerations into account.

Finally, it would be of interest to extend the literature on schemes that allow for distributed optimality guarantees in secondary frequency control. In this thesis, we have described two optimality schemes for secondary frequency regulation. In Chapter 6 we have considered an optimality scheme which allows for distributed stability and optimality guarantees for a broad range of power supply dynamics. However, this scheme requires knowledge of uncontrollable demand which in some cases might be difficult to obtain. Furthermore, as discussed in Chapter 2, the DAPI scheme also allows for distributed stability and optimality guarantees and requires only frequency measurements. However, existing results on DAPI schemes are restricted to proportional power supply dynamics which is restrictive. It would be interesting to investigate whether a scheme could allow for a sufficiently broad range of power supply dynamics with less requirements than the optimality scheme considered in Chapter 6. Such scheme could potentially overcome the main weaknesses of the two dominant schemes in literature and would allow for improved practical implementations.

These four ideas are the main research directions that follow from the work within this thesis. Other potential extensions are (i) the inclusion of voltage dynamics in the analysis, (ii) make use of subgradient techniques for secondary frequency control to allow for optimality when the vector fields are discontinuous, (iii) explore how market mechanisms might apply in the control of power supply via a price signal, possibly by making use of principles from game theory, (iv) study how dishonesty in the information provided between users might affect the behaviour of the power network and (v) consider switching between dynamic rather than piecewise constant schemes for controllable loads.

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